Stellar Evolution 2016 Q2

Ugeseddel 10 (week 48)

In the lectures on 1 December Günter finishes the discussion of asymptotic giant branch, including shell-source instabilities, thermal pulses and slow neutron-capture nucleosynthesis (*Kippenhahn, Weigert & Weiss*, Chapter 34), and the final stages of the evolution of massive stars. He then considers supernova explosions (*Kippenhahn, Weigert & Weiss*, Chapters 35 and 36). On 5 and 8 December he presents the compact objects that are the products of stellar evolution, i.e., white dwarfs and neutron stars (*Kippenhahn, Weigert & Weiss*, Chapters 37 and 38). On 12 December Amalie and Andreas will talk about the evolution of binary stars.

In the exercise class on 30 November we apply MESA to the limit of very low-mass stars, as discussed in *Lecture Notes on Stellar Structure and Evolution*, Exercise 10.1.

i) Consider the early evolution of a few stars of masses close to (both below and above) the claimed limiting mass of $0.08 M_\odot$. Consider the evolution of the central temperature with age, to locate the cases where this reaches a maximum and starts decreasing before proper nuclear burning starts. This can be defined as where 99% of the star's surface luminosity is obtained from nuclear reactions.

ii) Make your own determination of the minimum mass where a protostar develops stable nuclear burning. Investigate whether this depends on the abundance of heavy elements.

In addition, I suggest that you consider

iii) Exercise U10.1, below. This discusses in more detail the properties of the shell-burning instability.

29 November 2016 Jørgen Christensen-Dalsgaard

Exercise U10.1 Stability of a nuclear-burning shell

This exercise considers the stability of a shell source. The treatment is a little more detailed, and realistic, than presented in *Kippenhahn, Weigert & Weiss* (KWW), although some of the basic ideas are the same. It is based on the paper by Schwarzschild & Härm (1965), supplemented by ideas from Henyey & Ulrich (1972). A more realistic and complex (and hence less transparent) analysis, which included the case of two shell sources, was given by Defouw (1973). The notation follows KWW, except where otherwise noted.

We consider the behaviour of the shell. Conditions at the bottom of the shell are denoted by subscript "0" and are assumed to be fixed. Conditions at the top of the shell are denoted by subscript "1". "Δ" is used to denote differences between the bottom and the top of the shell (chosen so as to be positive), and "δ" denotes changes caused by the perturbation. The mass of the shell $\Delta m = m_1 - m_0$ is taken to be fixed. It is assumed that the shell is so thin that for some purposes variations across it can be neglected.

1) Show from the condition that the mass is unchanged that the average density perturbation $\delta\rho/\rho$ is related to the change in radius $\delta r_1/r_1$ at the top of the shell by

$$\frac{\delta\rho}{\rho} \approx -\left(2 + \frac{r_1}{\Delta r}\right)\frac{\delta r_1}{r_1}, \tag{1}$$

where $\Delta r = r_1 - r_0$ is the thickness of the shell. For a thin shell 2 can be neglected compared with $r_1/\Delta r$ and we recover the expression from KWW:

$$\frac{\delta\rho}{\rho} \approx -\frac{r_1}{\Delta r}\frac{\delta r}{r}, \tag{2}$$

where we dropped the subscripts "1" in $\delta r/r$ (note that homology corresponds to taking $\Delta r/r_1 = 1$ in equation (1), and hence to replacing $r_1/\Delta r$ by 3 in equation (2); see KWW). In the following we use equation (2).

2) Show that the pressure change over the shell is (again assuming the shell to be thin)

$$\Delta P = P_0 - P_1 = \frac{G\Delta m\, m_1}{4\pi r^4}, \tag{3}$$

and that therefore the pressure perturbation is given by

$$\frac{\delta P}{P} = -4\frac{\Delta P}{P}\frac{\delta r}{r}. \tag{4}$$

This relation reduces to the usual homology relation if $\Delta P/P = 1$. Given that it is not precisely clear at what level the reference P is measured, this is probably the most reasonable approximation if the shell is not very thin.

To simplify the notation we introduce

$$\Delta_P = \frac{\Delta P}{P}; \qquad \Delta_r = \frac{\Delta r}{r}; \qquad \Delta_T = \frac{\Delta T}{T}. \tag{5}$$

We assume that the equation of state gives, as usual,

$$\frac{\delta\rho}{\rho} = \alpha\frac{\delta P}{P} - \delta\frac{\delta T}{T}, \tag{6}$$

with $\alpha = \delta = 1$ for the ideal-gas case.

3) Show that

$$\frac{\delta r}{r} \approx -\frac{\delta}{4\alpha\Delta_P - {\Delta_r}^{-1}}\frac{\delta T}{T}, \tag{7}$$

$$\frac{\delta P}{P} \approx \frac{4\Delta_P\delta}{4\alpha\Delta_P - {\Delta_r}^{-1}}\frac{\delta T}{T}, \tag{8}$$

$$\frac{\delta\rho}{\rho} \approx \frac{{\Delta_r}^{-1}\delta}{4\alpha\Delta_P - {\Delta_r}^{-1}}\frac{\delta T}{T}. \tag{9}$$

Hence show from the first law of thermodynamics that the heat perturbation per unit mass is

$$\delta q = c_P T\left(1 - \nabla_{\rm ad}\frac{4\Delta_P\delta}{4\alpha\Delta_P - {\Delta_r}^{-1}}\right)\frac{\delta T}{T}. \tag{10}$$

As in KWW, this equation defines an effective specific heat, taking into account the hydrostatic reaction of the layer; for a thin layer, the pressure readjustment is small, and the layer behaves like a normal gas, getting warmer when heated up. The converse is true if the layer is sufficiently thick. Thus, broadly speaking, a thin layer is unstable and a thick layer is stable, given the rapid increase in nuclear reaction rates with increasing temperature.

To get a more secure estimate of the stability of the layer, however, the energetics of the perturbation must be taken into account, particularly the heat loss. We use the idealized temperature profile in the unperturbed shell illustrated in Figure 1a, and the particular form of the temperature perturbation shown in Figure 1b; note that δT is a measure of the average temperature perturbation over the shell, and hence the preceding relations for δP, δq, etc., must be supposed to hold in an average sense.

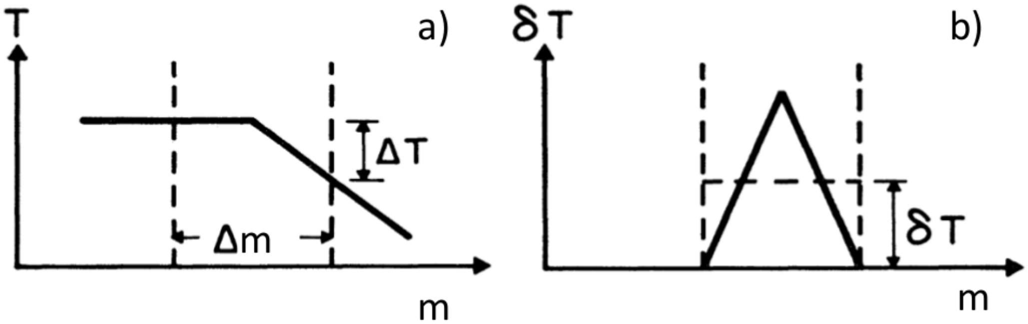

Figure 1: Schematic unperturbed temperature profile (a) and temperature perturbation (b). Adopted from Schwarzschild & Härm (1965).

4) Show that the unperturbed luminosity at the top of the shell is

$$l = 2(4\pi r^2)^2 \frac{4ac}{3} \frac{T^3}{\kappa} \frac{\Delta T}{\Delta m}. \tag{11}$$

Hence argue that the luminosity perturbations at the top and the bottom of the layer are approximately

$$l_+ = 2l \frac{T}{\Delta T} \frac{\delta T}{T}, \qquad l_- = -2l \frac{T}{\Delta T} \frac{\delta T}{T}. \tag{12}$$

5) Show from these relations, and the condition that the unperturbed shell is in equilibrium, that the heat perturbation per unit mass and time is

$$\frac{\mathrm{d}\delta q}{\mathrm{d}t} = \epsilon \left(\frac{\delta \epsilon}{\epsilon} - 4{\Delta_T}^{-1} \frac{\delta T}{T} \right). \tag{13}$$

6) We neglect the dependence of ϵ on ρ (argue that this is reasonable). Show that the energy equation may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\delta T}{T} \right) = (t^*)^{-1} \frac{\delta T}{T}, \tag{14}$$

where

$$t^* = t_{\mathrm{KH}} \frac{1 - \nabla_{\mathrm{ad}} \dfrac{4\Delta_P \delta}{4\alpha\Delta_P - {\Delta_r}^{-1}}}{\epsilon_T - 4{\Delta_T}^{-1}}, \tag{15}$$

$\epsilon_T = (\partial \ln \epsilon / \partial \ln T)_\rho$, and $t_{\mathrm{KH}} = c_P T \Delta m / l$ is the Kelvin-Helmholz time for the shell.

7) Taking for simplicity $\Delta_P = 1$, and assuming the ideal gas law, discuss the behaviour of t^* and hence the stability of the shell as its thickness is varied. What is the physical mechanism stabilizing a very thin shell?

8) Try to estimate what effect strong radiation pressure has on the stability of the shell, by evaluating t^* in the limit $\beta \to 0$, assuming again $\Delta_P = 1$. (Note that β enters into α, δ and c_P.)

References.

Defouw, R. J., 1973. [A simplified model for oscillatory secular modes]. *Astrophys. J.*, **182**, 215 – 224.

Henyey, L. G. & Ulrich, R. K., 1972. [Studies in stellar evolution. X. Hydrostatic adjustment]. *Astrophys. J.*, **173**, 109 – 120.

Schwarzschild, M. & Härm, R., 1965. [Thermal instability in non-degenerate stars]. *Astrophys. J.*, **142**, 855 – 867.