White dwarfs

Thermal properties and Evolution

- degenerate e⁻ in WD core provide high thermal conductivity, which, together with low L produce a very small ∇T, i.e. **degenerate core is essentially isothermal**.

- outermost, less-degenerate layers, however, where ρ is small, dominant energy transport is provided by radiation or convection, which is much less efficient than conduction.

- we therefore can assume a non-degenerate outer layer, in which T drops very sharply, isolating the degenerate, isothermal interior from outer space; we simplify this even further by assuming a **discontinuous transition** from degeneracy to non-degeneracy (ideal gas) at point with **subscript 0**.

- outer layers: diffusion approximation to radiative transfer: $F = -K\nabla T$

from: $\nabla_{\rm rad} = \dfrac{3}{16\pi acG}\dfrac{\kappa l P}{mT^4}$ and using Kramer's opacity law (bf & ff), $\kappa = \kappa_0 P T^{-4.5}$

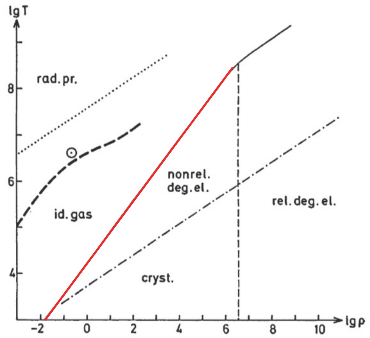
(KWW: 11.3.1)

$\longrightarrow T^{8.5} = BP^2\,;\quad B = 4.25\dfrac{3\kappa_0}{16\pi acG}\dfrac{L}{M}\,;$ replacing P by $\Re\varrho T/\mu \longrightarrow \boxed{\varrho = B^{-1/2}\dfrac{\mu}{\Re}T^{3.25}}$

transition point (0) is where (non-rel.) degenerate e⁻ pressure = pressure of ideal gas:

$$\boxed{\varrho_0 = C_1^{-3/2}T_0^{3/2}\,;}\quad C_1 = 1.207\times10^5\frac{\mu}{\mu_{\rm e}^{5/3}}\ {\rm cgs}$$

REMINDER: **The equation of state of stellar matter**



Border: ideal – degenerate (non-rel.)

$$\frac{\Re}{\mu}\varrho T = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_{\rm e}}\left(\frac{\varrho}{\mu_{\rm e}m_{\rm u}}\right)^{5/3}$$

$$\downarrow$$

$$\frac{T}{\varrho^{2/3}} = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_{\rm e}\Re m_{\rm u}^{5/3}}\frac{\mu}{\mu_{\rm e}^{5/3}}$$

$$= 1.207\times10^5\frac{\mu}{\mu_{\rm e}^{5/3}}$$

White dwarfs

Thermal prope

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- outer layers: diffusion approximation to r

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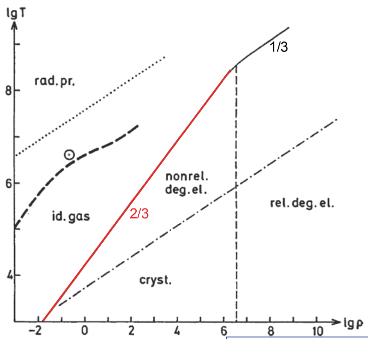

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$$\boxed{\varrho_0 = C_1^{-3/2}T_0^{3/2}\,;}\quad C_1 = 1.207\times10^5\frac{\mu}{\mu_{\rm e}^{5/3}}\ {\rm cgs} \longrightarrow T_0^{3.5} = \frac{B}{C_1^3}\left(\frac{\Re}{\mu}\right)^2 = \vartheta\frac{L/L_\odot}{M/M_\odot}\,.$$

White dwarfs

Thermal properties and Evolution

- for a typical composition & value for κ_0 :

$$T_0 \approx \vartheta^{2/7}\left(\frac{L/L_\odot}{M/M_\odot}\right)^{2/7} \approx \left(\frac{L/L_\odot}{M/M_\odot}\right)^{2/7} 5.9\times10^7\ {\rm K}\,.$$

This (rather simple) relation between L and T_0 will be important for deriving **cooling time** for WD.

For M =1M$_\odot$ and typical values L=(10^{-4}…10^{-2})L$_\odot \rightarrow T_0 \approx$ 4.2 …16x10^6 K (= isothermal T in interior)

$\rightarrow \rho_0 \approx 10^3$ gcm^{-3} (<< ρ_c).

Estimate of radial extension R-r_0 of non-degenerate envelope from T-r (τ) relation (KWW Ch.11.3.4):

$$T - T_{\rm eff} = f\left(\frac{R}{r}-1\right)\quad\text{with}\quad f := \nabla\frac{G\mu}{\mathcal{R}}\frac{M}{R}\quad;\quad \nabla := \frac{{\rm d}\ln T}{{\rm d}\ln P}$$

$$\longrightarrow \frac{R-r_0}{r_0} \approx \frac{\Re T_0}{\mu\nabla}\frac{R}{GM} \approx 0.82\frac{\overbrace{R/R_\odot}^{0.01}}{\underbrace{M/M_\odot}_{1}}\frac{T_0}{10^7\ {\rm K}}\,.\qquad (\mu = 4/3,\ \nabla = 0.4)$$

$\longrightarrow$ non-degenerate envelope <1%R $\longrightarrow$ WD radius well approximated by integrating over whole degeneracy as long as $T_{\rm eff}$ is small (i.e. $T_{\rm eff} \approx 0$ = cold EOS approximation).

Note: WD with more massive envelopes & higher T have larger radii up to 50%.

White dwarfs
Thermal properties and Evolution

- relatively high internal $T = 10^6 \ldots 10^7$ K sets limit to possible H - content in interior. For typical average values T=5x10^6 K & ρ=10^6 gcm^{-3} we obtain (pp chain)

$$\varepsilon_{pp} = 2.57 \times 10^4 \psi f_{11} g_{11} \varrho X_1^2 T_9^{-2/3} \mathrm{e}^{-3.381/T_9^{1/3}} \approx \underline{5 \times 10^4 \boxed{X_{\mathrm{H}}^2} \,\mathrm{erg\ g^{-1}\ s^{-1}}}$$
$$g_{11} = (1 + 3.82 T_9 + 1.51 T_9^2 + 0.144 T_9^3 - 0.0114 T_9^4)$$

- for which L would be for $M = 1 M_\odot$:

$$L/L_\odot \approx \frac{M_\odot}{L_\odot}\varepsilon_{pp} \approx 2.5 \times 10^4 X_{\mathrm{H}}^2 \,,$$

i.e. an observed luminosity $L \leq 10^{-3} L_\odot$ allows only $\underline{\boxed{X_{\mathrm{H}} \lesssim 2 \times 10^{-4}.}}$

Moreover, (secular) stability considerations (in degenerate matter; KWW Ch. 25.3.5)

$$\frac{d\vartheta_c}{dt} = \frac{l_s \varepsilon_T}{m_s T_c c^*}\vartheta_c := \frac{1}{D}\vartheta_c \quad \text{with} \quad c^* = c_P\left(1 - \nabla_{\mathrm{ad}}\frac{4\delta}{4\alpha - 3}\right) \quad ; \ \vartheta := \mathrm{d}T/T \ ; dT = dq/c^*$$

would show unstable burning (thermal runaway), because $c^* > 0$ in degenerate matter (δ=0), i.e. main contribution to luminosity of normal WD cannot be generated by thermonuclear reactions, as pointed out first by L. Mestel (1952): stable burning in nearly cold configurations only by pycnonuclear reactions near T=0.

White dwarfs
Thermal properties and Evolution

- so, if there are no thermonuclear reactions, then what energy sources supply the energy losses by radiation in a normal WD?

- to address this question we make use of the **virial theorem** (KWW Ch 3.1)

$$\zeta \dot{E}_{\mathrm{i}} + \dot{E}_{\mathrm{g}} = 0.$$

Potential energy in the gravitational field is (E_g < 0)

$$E_{\mathrm{g}} := -\int_0^M \frac{Gm}{r} dm$$

and the (total) internal energy $E_{\mathrm{i}} = E_{\mathrm{e}} + E_{\mathrm{ion}}$ consists of contributions from e$^-$ and ions;

ζ is an average of ζ'

$$\zeta' u = 3\frac{P}{\rho} \quad ; \ \zeta' = 1, \ \ldots, \ 2$$

non-relativistic case

relativistic case

where u is the internal energy per unit mass.

White dwarfs
Thermal properties and Evolution

The **total energy** is then $W = E_{\mathrm{i}} + E_{\mathrm{g}}$, and the energy equation $L = -\mathrm{d}W/\mathrm{d}t$ together with the virial theorem leads to

$$L = -\dot{W} = -\frac{\zeta - 1}{\zeta}\dot{E}_{\mathrm{g}} = (\zeta - 1)\dot{E}_{\mathrm{i}} \,,$$

i.e. L>0 (loss of energy) requires contraction, $\dot{E}_{\mathrm{g}} < 0$, and an increase of internal energy $\dot{E}_{\mathrm{i}} > 0$, which is also the case for (non-degenerate) stars.

In normal stars (both e$^-$ and ions are non-degenerate), loss of energy (L>0) leads to heating $\dot{T} > 0$, which corresponds to a negative gravothermal heat c^* < 0 (total energy is negative: -E_g/2); i.e. ½ of E_g is radiated away & the other ½ to **heat the gas**, i.e. to increase E_i=E_e+E_{ion} ~ T.

In WD e$^-$ are degenerate and an increase of E_i leads to a different distribution between E_e and E_{ion} such that ions release about as much thermal energy **by cooling** as the WD loses by radiation.

We have (with $\zeta = 2$): $L = -\dot{E}_{\mathrm{g}}/2$ & from $-E_{\mathrm{g}} \sim 1/R \sim \varrho^{1/3}$ $\longrightarrow \dot{E}_{\mathrm{g}}/E_{\mathrm{g}} = (1/3)\dot{\varrho}/\varrho.$

compression increases Fermi energy E_F of e$^-$: $E_{\mathrm{e}} \approx E_{\mathrm{F}} \sim p_{\mathrm{F}}^2 \sim \varrho^{2/3} \longrightarrow \dot{E}_{\mathrm{e}}/E_{\mathrm{e}} = (2/3)\dot{\varrho}/\varrho.$

$$E_{\mathrm{g}} = -2E_{\mathrm{i}}$$
$$\longrightarrow \quad \dot{E}_{\mathrm{e}} \approx 2\frac{E_{\mathrm{e}}}{E_{\mathrm{g}}}\dot{E}_{\mathrm{g}} = -\frac{E_{\mathrm{e}}}{E_{\mathrm{i}}}\dot{E}_{\mathrm{g}} \,.$$

White dwarfs
Thermal properties and Evolution

If WD is already cool: $E_{\mathrm{ion}} \ll E_{\mathrm{e}} \ \rightarrow \ E_{\mathrm{i}} = E_{\mathrm{ion}} + E_{\mathrm{e}} \approx E_{\mathrm{e}} \ \rightarrow \ \dot{E}_{\mathrm{e}} \approx -\dot{E}_{\mathrm{g}} = 2L$
(P_e dominates)

$\longrightarrow$ nearly all the energy released by contraction is used to raise E_F of e$^-$!

With $\dot{E}_{\mathrm{e}} \approx -\dot{E}_{\mathrm{g}}$ the energy balance $L = -\dot{E}_{\mathrm{ion}} - \dot{E}_{\mathrm{e}} - \dot{E}_{\mathrm{g}}$ becomes:

$$L \approx -\dot{E}_{\mathrm{ion}} \sim -\dot{T} \,.$$

$\longrightarrow$ Ions release about as much thermal energy **by cooling** as the WD loses by radiation!

The contraction is then seen to be the consequences of the decreasing ion pressure.

Note that in spite of the decreasing E_{ion}, the internal energy E_i rises, since $\dot{E}_{\mathrm{ion}} + \dot{E}_{\mathrm{e}} \approx L.$

The evolution tends finally to a cold black dwarf; then the contraction has stopped and all of the internal energy E_i is in the form of Fermi energy E_F of the e$^-$.

White dwarfs
Simple theory of WD cooling

We start with the energy released by gravitational contraction (need to include all time-dependence):

$$\varepsilon_g = -c_v\dot{T} + \frac{T}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_v \dot{\rho} - \underbrace{\left(\frac{\partial E_i}{\partial X_0}\right)_{M_s}\frac{dX_0}{dt}}_{\text{energy released by chemical re-adjustment}} + q_s\dot{M}_s$$

energy released by chemical re-adjustment, stored in form of chemical potentials; X_0 is abundance by mass of heavier component (O) of the WD core, consisting of two species, e.g. C & O; M_s is mass at boundary of solid core.

Contribution from the latent heat q_s, which is generally small (<5% of total L).

energy releases per gram of crystallized matter due to change in X (Isern et al. 1997):

$$\varepsilon_g^{\rm cryst} = -\left(\frac{\partial E_i}{\partial X_0}\right)_{M_s}\frac{dX_0}{dt} \simeq -\left(X_0^{\rm sol} - X_0^{\rm liq}\right)\left[\left(\frac{\partial E_i}{\partial X_0}\right)_{M_s} - \left\langle\frac{\partial E_i}{\partial X_0}\right\rangle\right]$$

energy released in crystallized layer as a result of the changing concentration of oxygen X_o.

energy absorbed in convective region ΔM driven by Rayleigh-Taylor instability, where

$$\left\langle\frac{\partial E_i}{\partial X_0}\right\rangle = \frac{1}{\Delta M}\int_{\Delta M}\left(\frac{\partial E_i}{\partial X_0}\right)dm\,.$$

White dwarfs
Simple theory of WD cooling

We start with the energy released by gravitational contraction (time-dependent terms):

$$\varepsilon_g = -c_v\dot{T} + \frac{T}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_v \dot{\rho}$$

Integration over whole star (mass), neglecting not only ε_{nuc} & ε_ν, but also volume change (last term)

$$-L \approx \int_0^M c_v\dot{T}\,dm \approx c_v\dot{T}_0 M\,,$$

where we assumed an isothermal interior with T=T_0.

For ideal ion gas: $c_v^{\rm ion} = \frac{3}{2}\frac{k}{Am_u}\,.$

For degenerate e$^-$ gas (Chandrasekhar 1939):

$$c_v^{\rm el} = \frac{\pi^2 k^2}{m_e c^2}\frac{Z}{Am_u}\frac{\sqrt{1+x^2}}{x^2}T \qquad [x = p_F/m_e c]$$

$$\approx \frac{\pi^2 k}{2}\frac{Z}{Am_u}\frac{kT}{E_F}\,, \quad \text{for } x \ll 1\,.$$

The ratio (for x<<1) $\frac{c_v^{\rm el}}{c_v^{\rm ion}} = \frac{\pi^2}{3}Z\frac{kT}{E_F}$ is rather small for small values of kT/E_F and Z.

White dwarfs
Simple theory of WD cooling

We therefore approximate $c_v \approx c_{ion}$, and the energy equation describing L is determined by the change of the internal energy of the ions only.

We now insert $-L \approx c_v\dot{T}_0 M \longrightarrow T_0^{3.5} = \frac{B}{C_1^3}\left(\frac{\Re}{\mu}\right)^2 = \vartheta\frac{L/L_\odot}{M/M_\odot}$

T_0 is T @ transition point where P_e=P_{ideal} of radiative (non-degenerate) outer layers.

and obtain $$\dot{T} = -\frac{L_\odot}{M_\odot}\frac{1}{c_v\vartheta}T^{7/2}\,.$$

This equation can now be integrated from t =0 (when T was much higher) to the present time t = τ:

$$\tau = \frac{2}{5}\frac{M_\odot}{L_\odot}c_v\vartheta T^{-5/2} = \frac{2}{5}c_v\frac{MT}{L}$$

$$= \frac{2}{5}\left(\frac{M_\odot}{L_\odot}\vartheta\right)^{2/7} c_v\left(\frac{M}{L}\right)^{5/7} \approx \frac{4.7\times10^7\,\text{years}}{A}\left(\frac{M/M_\odot}{L/L_\odot}\right)^{5/7}.$$

This is the cooling time τ, as obtained first by L. Mestel (1952).

White dwarfs
Simple theory of WD cooling

Cooling time τ (L. Mestel 1952):

$$\tau \approx \frac{4.7\times10^7\,\text{years}}{A}\left(\frac{M/M_\odot}{L/L_\odot}\right)^{5/7}$$

For A=4, M=1M$_\odot$ & L=10^{-3} L$_\odot$ ⟶ $\tau \approx 10^9$ y .

For CO-WDs, A≈14, M=1M$_\odot$ & L=10^{-4} L$_\odot$ ⟶ $\tau \approx 2 \times 10^9$ y .

Note that τ increases for massive WDs and the lighter the main element is.

Also, the specific heat c_v is also very important: larger values will result in longer cooling times τ.

Moreover, at small M (ρ) & large T (and Z) the electron contribution $c_v^{\rm el}$ becomes important, e.g., for T=10^7 K , M=0.5M$_\odot$ and a CO core we get

from $\frac{c_v^{\rm el}}{c_v^{\rm ion}} = \frac{\pi^2}{3}Z\frac{kT}{E_F} \longrightarrow \boxed{c_v^{\rm el} \approx 0.25 c_v^{\rm ion}}$

White dwarfs

Simple theory of WD cooling: **the effect of c_v**

For small T ions dominate completely: $c_v = c_v^{\rm ion}$, but c_v is influenced by crystallization.

Properties of ions depend on two dimensionless quantities:

(i) $$\Gamma_{\rm c} \;=\; \frac{(Ze)^2}{r_0 kT} \;\simeq\; 2.7\times10^{-3}\;\frac{Z^2\, n_{\rm ion}^{1/3}}{T}\;,$$

(ii) $$\frac{T}{\Theta}\;,$$ where Θ is the Debye temperature defined by

$$k\Theta = \hbar\Omega_{\rm p}\;,\quad \Omega_{\rm p} = \frac{2Ze}{Am_{\rm u}}(\pi\varrho)^{1/2}\;,$$

characteristic energy of lattice oscillations

ion plasma frequency

$$\longrightarrow\quad \Theta = \frac{he}{km_{\rm u}\sqrt{\pi}}\frac{Z}{A}\varrho^{1/2} \approx 7.8\times10^3\ {\rm K}\cdot\frac{Z}{A}\varrho^{1/2} \qquad (\varrho \text{ in g cm}^{-3}).$$

White dwarfs

Simple theory of WD cooling: **the effect of c_v**

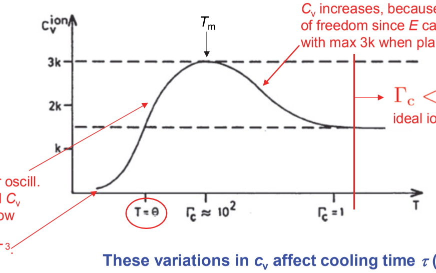
Debye temperature:

$$\Theta = \frac{he}{km_{\rm u}\sqrt{\pi}}\frac{Z}{A}\varrho^{1/2} \approx 7.8\times10^3\ {\rm K}\cdot\frac{Z}{A}\varrho^{1/2}$$

lattice melting temperature

Lattice oscillations only possible if $\Theta < T_{\rm m}$, a condition which is met in typical CO WDs.

$\Gamma_{\rm c} >> 1$

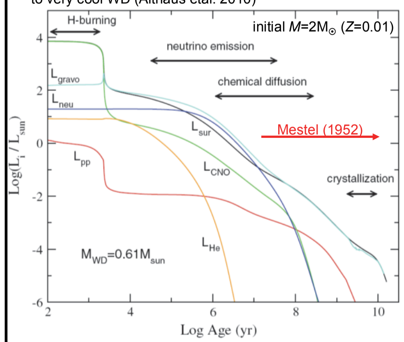


These variations in c_v affect cooling time τ (increase by ~10%).

White dwarfs

Modelled WD cooling

Cooling of CO WD from PN stage to very cool WD (Althaus etal. 2010)



$L_{\rm surf}$ … surface luminosity (losses)
$L_{\rm pp}$ … H-burning (pp)
$L_{\rm CNO}$ … H-burning (CNO)
$L_{\rm He}$ … He burning
$L_{\rm neu}$ … neutrino losses
$L_{\rm gravo}$ …release of thermal (ion) and gravitational potential

Early stage: $L_{\rm surf}$ dominated by CNO burning (pp essentially negligible everywhere) and $L_{\rm grav}$ balances $L_{\rm neu}$ losses.

After ~10^4y: H-shell extinguishes rapidly; WD begins to cool & gets fainter.

Few 10^5y: $L_{\rm neu}$ losses more important than photon emission for cooling ($L_{\rm neu}$~$5L_{\rm surf}$), and energy results completely from gravothermal energy $L_{\rm grav}$, gained from internal energy of the core.

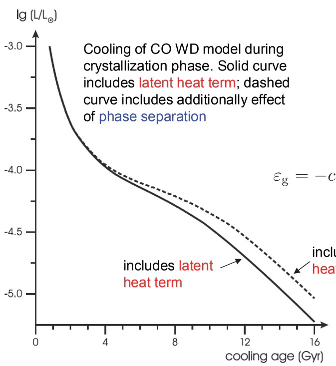
At ~ 10^7y: $L_{\rm neu}$ sink fades away and WD is now in the phase where Mestel's cooling law applies; **thermal energy of ions only energy source!**

At ~ 10^9y: crystallization and convection sets in; this phase is shown in the next figure.

$L_{\rm He}$ is never an important energy source.

White dwarfs

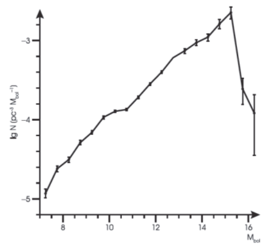
Modelled WD cooling



$$\varepsilon_g = -c_{\rm v}\dot{T} + \frac{T}{\rho^2}\left(\frac{\partial P}{\partial T}\right)_{\rm v}\dot{\rho} - \left(\frac{\partial E_{\rm i}}{\partial X_0}\right)_{M_{\rm s}}\frac{{\rm d}X_0}{{\rm d}t} + q_{\rm s}\dot{M}_{\rm s}$$

White dwarfs

Modelled WD cooling



Cooling time can be connected with observed number of WD as a function of *L* (M_{bol}).

Since WD evolution slows down with time, it is to be expected that the number of observed WDs is increasing with decreasing luminosity (luminosity function).

Then, the luminosity function should show a sudden cutoff indicating the age of the oldest WDs and thus the oldest stars in the observed sample.

Age of galactic disk in solar neighbourhood: ~ 8 Gy .

Fig. 37.8. Luminosity function of white dwarfs obtained from 6000 objects in the *Sloan Digital Sky Survey*, data release 3 (Abazajian et al. 2005), including error bars of the star countings, which are significant after the peak of the luminosity function, due to the extreme faintness of the objects. The last few data points at and after the break of the luminosity function also depend somewhat on the assumption about the interior carbon and oxygen abundances. The break at $M_{\rm bol} = 15.3$, or equivalently $\log L/L_\odot = -4.3$, could correspond to an age of the coolest WDs of 8 Gyr, or be a few Gyr higher (Kilic et al., 2010, and Salaris, private communication). Data for this figure were taken from Harris et al. 2006; Fig. 7.