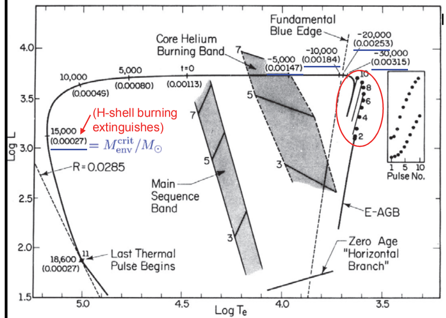
White dwarfs (progenitor)



Iben (1985; Ql. J. R. astr. Soc. 26, 1)
[evolution of a $2M_\odot$ star, losing ~ $1.2M_\odot$ after AGB phase via ordinary wind & PN]

(thermal) pulses are more or less an envelope phenomenon with *no influence on core*.

inner part of C-O core resembles more and more a *white dwarf*.

H-rich envelope, small in M but thick in R, gives appearance of a red giant.

however, during thermal pulses envelope **loses mass** from surface, which is ejected into space; additional **mass loss** due to H-burning @ bottom.

envelope shrinks and within ~ 10^4 y star moves to the left (WD-track).

An originally $2M_\odot$ star develops a C-O core of ~ $0.6M_\odot$; Planetary Nebula (PN) ejection (superwind) occurs at ~ the 10th pulse. When $T_{\rm surf}$ of star reaches ~ 30 000K (@t=0), photons emitted from surface are 'hard' enough to ionize surrounding nebular shell. The star is then called a Planetary Nebular Nucleus (PNN).

H-shell burning extinguishes ($M_{\rm env}$ drops below $0.00027M_\odot$) and the star becomes a white dwarfs (WD).

White dwarfs (progenitor)

Evolution of 3.5 $M_\odot$ star

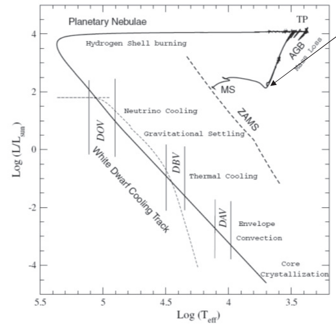


Fig. 7 Hertzsprung–Russell diagram for the full evolution of a 3.5 $M_\odot$ star from the ZAMS to the white dwarf domain. Mass-loss episodes at the thermally pulsing AGB reduce the stellar mass to 0.66 $M_\odot$. The various physical processes which occur as white dwarfs cool as well as the domain of the pulsating instability strips for the DOV, DBV, and DAV are indicated. *Thin dashed line* indicates the neutrino luminosity

Althaus et al. (2010, A&ARv 18, 471)

Core He-burning is ignited in non-degenerate core, which 'stops' core contraction (envelope expansion).

The C-O core characterizing the later emerging WD remnant is built up.

He burning stops in central core if ^{4}He is completely processed to ^{12}C, ^{14}O, & ^{20}Ne . Burning continues in a shell around the exhausted core: evolution along AGB.

While He shell burns outwards, the C-O core increases in mass and contracts. Star has now two shell sources (He & H).

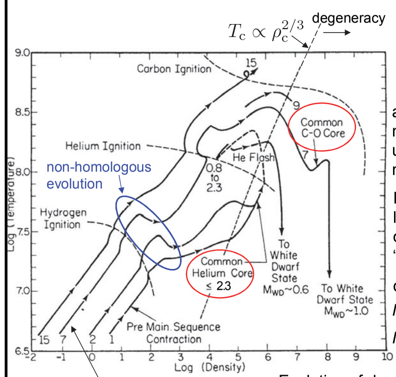
He-shell source becomes unstable → TP.

When envelope mass is reduced to < 10^{-3} $M_\odot$ remnant star moves rapidly to left (PN).

When H-envelope mass < 10^{-4} $M_\odot$, H-shell burning essentially extinct & surface L drops rapidly → WD left with only gravitational and thermal energy sources.

Later Phases

Evolution of central regions



$$T_c \propto \rho_c^{1/3}$$
$$\mathrm{d}\ln T_c = (4\alpha - 3)/(3\delta)\,\mathrm{d}\ln\rho$$

standard evolution computations suggest the following approximate limiting masses:

M < 2.3 $M_\odot$ develop degenerate He cores

M ≤ 9 $M_\odot$ develop degenerate C-O cores

M > 9 $M_\odot$ C-O cores remain non-degenerate

after star developed degenerate core it has not necessarily reached very end of nuclear life, unless shell-source cannot sufficiently increase mass of degenerate core.

If shell source can increase core mass to at least to critical core mass M'_c, next burning is only delayed, and will be ignited in a later "flash".

critical masses for ignition in degenerate core:

M'_c > 0.48 $M_\odot$ ignition in degenerate He core

M'_c > 1.40 $M_\odot$ ignition in degenerate C-O core

Evolution of degenerate C-O core similar to degen. He core. Core structure independent of envelope → evolution of central values converges for stars of different M but with same M_c.

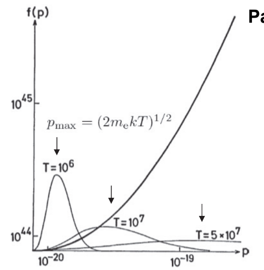
REMINDER: (HUSK)

Degenerate Electron Gas

W. Pauli: each quantum cell (x, y, z, p_x, p_y, p_z) can only hold **2 electrons**

quantum cell volume: $\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z\mathrm{d}V = h^3$

number of electrons in shell $[p,\ p+\mathrm{d}p]$: $8\pi p^2\mathrm{d}p\mathrm{d}V/h^3$



Pauli: $f(p)\mathrm{d}p\mathrm{d}V \le \frac{8\pi p^2\mathrm{d}p\mathrm{d}V}{h^3}$

$T\downarrow$ → $p_{\max}$ to smaller p values

$n_e\mathrm{d}V = \mathrm{d}V\int_0^\infty f(p)\mathrm{d}p$ = constant

violation also for T constant and high densities, since $f(p)\mathrm{d}p \sim n_e$

need to include quantum effects if either T too low or electron density too high, i.e. if electrons become *degenerate*.

REMINDER: (HUSK)

Degenerate Electron Gas

The **completely degenerate electron gas**

Non-relativistic limit: $x \to 0 \quad (n \to 3/2)$ $\qquad x \equiv \frac{p_F}{m_e c}$ ..... importance of relativistic effects

$$f(x) \to \frac{8}{5}x^5\,, \quad g(x) \to \frac{12}{5}x^5\,, \quad \text{for} \quad x \to 0$$

$$P_e = \frac{8\pi m_e^4 c^5}{15h^3}x^5 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_e m_u^{5/3}}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$

$$P_e = 1.0036 \times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{5/3} \text{(cgs)}$$

Independent of *T*

$$U_e = \frac{3}{2}P_e$$

REMINDER: (HUSK)

Degenerate Electron Gas

The **completely degenerate electron gas**

Extreme relativistic limit: $x \to \infty \quad (n \to 3)$

$$f(x) \to 2x^4\,, \quad g(x) \to 6x^4\,, \quad \text{for} \quad x \to \infty$$

$$P_e = \frac{2\pi m_e^4 c^5}{3h^3}x^4 = \left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{8m_u^{4/3}}\left(\frac{\rho}{\mu_e}\right)^{4/3}$$

$$P_e = 1.2435 \times 10^{15}\left(\frac{\rho}{\mu_e}\right)^{4/3} \text{(cgs)}$$

Independent of *T*

$$U_e = 3P_e$$

White dwarfs

Core consists of degenerate matter, i.e. the mechanical properties (momentum & mass equ.) are more or less decoupled from the thermal (energy equ.) properties (polytropes!)

Chandrasekhar's Theory

- treats mechanical structure of WD
- assumptions: (i) pressure *P* determined only from ideal (non-interacting) degenerate e$^-$, but of arbitrary **degree of relativity**

$$x \equiv \frac{p_F}{m_e c} \propto \rho^{1/3} \text{....} \quad \text{relativity parameter}$$

(ii) ions provide mass ($\rho = \mu_0 m_u n$).

EOS (completely degenerate, ideal electrons e$^-$):

$$P_e = \frac{\pi c^5 m_e^4}{3h^3}f(x) \quad = C_1\, f(x) \qquad f(x) = x(2x^2-3)(x^2+1)^{1/2} + 3\ln\left[x+(1+x^2)^{1/2}\right]$$

$$\rho = \mu_e m_u \frac{8\pi m_e^3 c^3}{3h^3}x^3 \quad = C_2 x^3$$

White dwarfs

Chandrasekhar's Theory

As in the discussion of the polytropes, we start with **Poisson's equation**

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 4\pi G\varrho\,.$$

and eliminate the gradient of the potential with the hydrostatic support equ.:

$$\frac{dP}{dr} = -\frac{d\Phi}{dr}\varrho$$

to obtain with the help of the previous expressions for *P* & ρ of a degenerate (rel.) e$^-$ gas

$$\frac{C_1}{C_2}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{x^3}\frac{df(x)}{dr}\right) = -4\pi G C_2 x^3\,.$$

df/dr can be obtained from differentiating the left-hand side of (KWW: 15.12) w.r.t. *x*

$$\longrightarrow \quad \frac{1}{x^3}\frac{df(x)}{dr} = 8\frac{d}{dr}[(x^2+1)^{1/2}] := 8\frac{dz}{dr}\,, \qquad \text{where} \qquad z^2 := x^2+1\,.$$

White dwarfs

Chandrasekhar's Theory

$$\frac{1}{x^3}\frac{df(x)}{dr} = 8\frac{dz}{dr} \longrightarrow \quad \frac{C_1}{C_2}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{x^3}\frac{df(x)}{dr}\right) = -4\pi G C_2 x^3$$

$$\downarrow$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dz}{dr}\right) = -\frac{\pi G C_2^2}{2C_1}(z^2-1)^{3/2}$$

As before (polytropes) we replace *r* and *z* by the dimensionless variables

$$\zeta := \frac{r}{\alpha}\,, \quad \alpha = \sqrt{\frac{2C_1}{\pi G}}\frac{1}{C_2 z_c}\,,$$

$$\varphi := \frac{z}{z_c}\,,$$

central value of z; $z^2 - 1 \propto \rho^{2/3}$

central value of *z*

White dwarfs

Chandrasekhar's Theory

with these dimensionless variables the 2nd-order DE becomes

$$\frac{1}{\zeta^2}\frac{d}{d\zeta}\left(\zeta^2\frac{d\varphi}{d\zeta}\right) = -\left(\varphi^2 - \frac{1}{z_c^2}\right)^{3/2}$$

Chandrasekhar's DE for WD structure.

$$\updownarrow$$

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{dw}{dz}\right) + w^n = 0\,.$$

Lane-Emden equation

these are different z!

Chandrasekhar's DE becomes the Lane-Emden equation for (polytropic) indices *n*=3 and *n*=3/2 in the limits $z \to \infty\ (x \to \infty)$ and $z \to 1\ (x \to 0)$ respectively.

White dwarfs

Chandrasekhar's Theory

$$\frac{d^2\varphi}{d\zeta^2} + \frac{2}{\zeta}\frac{d\varphi}{d\zeta} + \left(\varphi^2 - \frac{1}{z_c^2}\right)^{3/2} = 0$$

central conditions @ $\zeta = 0$: $\varphi = 1\,, \quad \varphi' = 0$

with these central conditions and with adopting a value for z_c, the 2nd-order DE can be integrated outwards (initial-value problem (IVP)).

Also, adopting a value for μ_e (in C_2, e.g. μ_e=2 for He and CO WDs), the density stratification is

$$\varrho = C_2 x^3 = C_2(z^2-1)^{3/2} = C_2 z_c^3\left(\varphi^2 - \frac{1}{z_c^2}\right)^{3/2}\,.$$

surface is reached @ $\zeta = \zeta_1$: $x_1 = 0\,, \quad z_1 = 1\,, \quad \varphi_1 = 1/z_c$
(where ρ =0)

surface radius R: $$R = \alpha\zeta_1 = \sqrt{\frac{2C_1}{\pi G}}\frac{1}{C_2 z_c}\zeta_1\,.$$

White dwarfs

Chandrasekhar's Theory

The mass *M* can be obtained from the mass conservation equation, replacing *r* by ζ and ρ by the expression given on the previous page:

$$\varrho = C_2 z_c^3\left(\varphi^2 - \frac{1}{z_c^2}\right)^{3/2} \qquad \zeta := \frac{r}{\alpha}$$

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \varrho\, dr \\ &= 4\pi\alpha^3 C_2 z_c^3 \int_0^{\varphi_1} \zeta^2\left(\varphi^2 - \frac{1}{z_c^2}\right)^{3/2} d\zeta \\ &= 4\pi\alpha^3 C_2 z_c^3 \left(-\zeta^2\frac{d\varphi}{d\zeta}\right)_1 \\ &= \frac{4\pi}{C_2^2}\left(\frac{2C_1}{\pi G}\right)^{3/2}\left(-\zeta^2\frac{d\varphi}{d\zeta}\right)_1\,, \end{aligned}$$

where for the integrand in the 2nd line we replaced the '( )'-term by the l.h.s. of

$$\frac{1}{\zeta^2}\frac{d}{d\zeta}\left(\zeta^2\frac{d\varphi}{d\zeta}\right) = -\left(\varphi^2 - \frac{1}{z_c^2}\right)^{3/2}\,.$$

White dwarfs

Chandrasekhar's Theory

Table 37.1. Numerical results of Chandrasekhar's theory of white dwarfs. Subscripts c and 1 refer to centre and surface, respectively. (After Cox & Giuli, 1968, vol. II, Chap. 25)

	$1/z_c^2$	x_c	ζ_1	$(-\zeta^2 d\varphi/d\zeta)_1$	ϱ_c/μ_e (g cm^{-3})	$\mu_e^2 M$ ($M_\odot$)	$\mu_e R$ (km)
fully relativistic →	0	∞	6.8968	2.0182	∞	5.84	0
	0.01	9.95	5.3571	1.9321	9.48×10^8	5.60	4.170
	0.02	7	4.9857	1.8652	3.31×10^8	5.41	5.500
	0.05	4.36	4.4601	1.7096	7.98×10^7	4.95	7.760
	0.1	3	4.0690	1.5186	2.59×10^7	4.40	10.000
	0.2	2	3.7271	1.2430	7.70×10^6	3.60	13.000
	0.3	1.53	3.5803	1.0337	3.43×10^6	2.99	16.000
	0.5	1	3.5330	0.7070	9.63×10^5	2.04	19.500
	0.8	0.5	4.0446	0.3091	1.21×10^5	0.89	28.200
non-relativistic →	1.0	0	∞	0	0	0	∞

↑ input ↑ $x_c = (z_c^2 - 1)^{1/2}$

Chandrasekhar limit M_{Ch}:

Chandrasekhar mass $M_{Ch} = \left(\frac{2}{\mu_e}\right)^2 \times 1.459 M_\odot = \frac{5.836}{\mu_e^2} M_\odot$

$z \to \infty$ $(x \to \infty)$: degenerate relativistic (n = 3) polytropes.

REMINDER: (HUSK)

Polytropes (simple stellar models)

Polytropic stellar models with fixed *K*

Example for non-relativistic, degenerate e⁻ gas (n=3/2):

$$P_e = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e} \frac{1}{(\mu_e m_u)^{5/3}} \rho^{5/3} = K \rho^\gamma \rightarrow \text{with } \mu_e \text{ fixed} \rightarrow K \text{ fixed}$$

[before: $K(n, A, \rho_c)$]

For any n (< 5 for finite radius):

$$A^{-2} = \left(\frac{r}{z}\right)^2 = \frac{1}{4\pi G}(n+1) K \varrho_c^{\frac{1-n}{n}} \longrightarrow A^{-1} = A^{-1}(\rho_c) \propto \rho_c^{\frac{1-n}{2n}} \quad \text{for given } n \text{ and } K.$$

with $R = z_n A^{-1} \rightarrow R \propto \rho_c^{\frac{1-n}{2n}}$

& with $M \propto \rho_c R^3 \rightarrow M = C_1 \varrho_c^{\frac{3-n}{2n}}$; $C_1 = 4\pi\left(-\frac{w'}{z}\right)_{z_n} z_n^3 \left(\frac{n+1}{4\pi G}\right)^{3/2} K^{3/2}$.

$\longrightarrow$ $R \propto M^{\frac{1-n}{3-n}}$.

i.e. for given n & K only a one-dimensional manifold of models, ***either*** parameter ***M or R***.

Before we had a two-dimensional manifold M and R when K was a free parameter.

REMINDER: (HUSK)

Polytropes (simple stellar models)

Chandrasekhar's limiting mass

For high densities (e.g., WD centres) e⁻ become ***ultra-relativistic*** (UR), for which n = 3 !

$$\text{for } n = 3 \rightarrow M = C_1 \text{ , i.e.} \quad M = 4\pi\left(-\frac{w'}{z}\right)_{z_3} z_3^3 \left(\frac{K}{\pi G}\right)^{3/2} .$$

$$K = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8(m_u \mu_e)^{4/3}} \longrightarrow M = 0.197 \left[\left(\frac{hc}{G}\right)^{3/2} \frac{1}{m_u}\right] \frac{1}{\mu_e^2}$$

Chandrasekhar mass $M_{Ch} = \frac{5.836}{\mu_e^2} M_\odot$ for degenerate relativistic polytropes.

This is the only possible mass for relativistic degenerate polytropes!

Indeed, no white dwarf has been found with $M > 1.46\, M_\odot$.

Note: in this UR limit degenerate pressure of (ideal) fermions is independent of rest mass, i.e. this limit is also valid for ideal gas of degenerate neutrons!

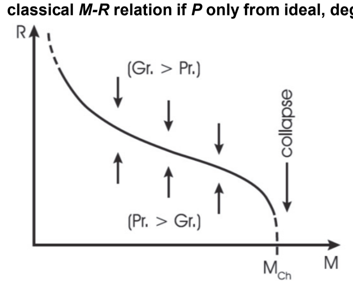
White dwarfs

Chandrasekhar's Theory

as for polytropes (e.g., n=3/2 for non-rel. degen. e⁻) $\mathrm{d}R/\mathrm{d}M < 0$ ($R \propto M^{\frac{1-n}{3-n}}$), but exponent of M is no longer constant.

$1/z_c^2$	x_c	$\mu_e^2 M$ ($M_\odot$)	$\mu_e R$ (km)
0	∞	5.84	0
0.01	9.95	5.60	4.170
0.02	7	5.41	5.500
0.05	4.36	4.95	7.760
0.1	3	4.40	10.000
0.2	2	3.60	13.000
0.3	1.53	2.99	16.000
0.5	1	2.04	19.500
0.8	0.5	0.89	28.200
1.0	0	0	∞

classical *M-R* relation if *P* only from ideal, degen. e⁻



Chandrasekhar mass $M_{Ch} = \left(\frac{2}{\mu_e}\right)^2 \times 1.459 M_\odot = \frac{5.836}{\mu_e^2} M_\odot$

White dwarfs

Chandrasekhar's Theory

The rather counter-intuitive relation $\mathrm{d}R/\mathrm{d}M < 0$ (i.e. iron spheres becomes smaller with mass) deserves some discussion:

assume rough averages in the hydrostatic support equation, $\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4}$, by replacing

the absolute value of dP/dm by P/M, and, m/r^4 by M/R^4 :

$$\underbrace{\frac{P}{M}}_{f_p} \approx \underbrace{\frac{GM}{4\pi R^4}}_{f_g}$$

we adopt EOS for degenerate gas $P \sim \varrho^\gamma \sim \left(\frac{M}{R^3}\right)^\gamma \rightarrow f_p \sim \frac{M^{\gamma-1}}{R^{3\gamma}}$; $f_g \sim \frac{M}{R^4}$.

For hydrostatic equilibrium the ratio $f := \frac{f_g}{f_p}$ has to be unity, i.e.:

$\mu_e = 2$ for CO WD
$\mu_e = 2.15$ for Fe WD

non-relativistic: $P_e \propto (\rho/\mu_e)^{5/3}$

$$f := \frac{f_g}{f_p} \sim M^{2-\gamma} R^{3\gamma-4} = \left\{ \begin{array}{ll} M^{1/3} R\,, & \text{for } \gamma = 5/3 \\ M^{2/3}\,, & \text{for } \gamma = 4/3 \end{array} \right\} = 1\,.$$

fully relativistic: $P_e \propto (\rho/\mu_e)^{4/3}$

White dwarfs

Chandrasekhar's Theory

$$f := \frac{f_g}{f_p} \sim M^{2-\gamma} R^{3\gamma-4} = \begin{cases} M^{1/3} R\,, & \text{for } \gamma = 5/3 \\ M^{2/3}\,, & \text{for } \gamma = 4/3 \end{cases}$$

e.g. for $M < M_{Ch}$ & $\gamma = 5/3$, the star easily finds equilibrium by adjusting R to suffice $f = 1$. If now M is increased, then $f > 1$ (gravity dominates over pressure), and R must decrease.

if, however, e^- are relativistic, i.e. $\gamma = 4/3$, f is independent of R, and equilibrium ($f = 1$) can only be achieved by adjusting M to a certain value M_{Ch}:

$M < M_{Ch} \rightarrow f < 1 \rightarrow$ dominant P makes star expand $\rightarrow$ e^- less relativistic $\rightarrow \gamma > 4/3$

$M > M_{Ch} \rightarrow f > 1 \rightarrow$ dominant *gravity* makes star contract: but this does not help and star collapses without finding equilibr.

M_{Ch} is mass limit for finding an equilibrium solution.

(Note: $\gamma < 4/3$ means dynamically unstable! KWW: Ch. 25.3.2)

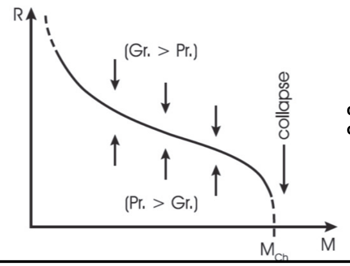
White dwarfs

Corrected mechanical structure (e.g., more realistic EOS)

- until now we assumed **ideal**, fully degenerate e^- gas.
- however, corrections near both ends of mass range are required:

(a) for $M \rightarrow 0$ (cold, or near cold configuration) we expect $R \rightarrow 0$, $\rho \approx$ constant (e.g., planets), but current solution provides $R \rightarrow \infty$, $\rho \rightarrow 0$, as a result of **missing electrostatic interactions**.

(b) @ limiting mass $M \rightarrow M_{Ch}$ (@ high densities) simple theory gives $\rho \rightarrow \infty$, but we need to include effects of the weak interaction (**inverse β decay**) and the possibility of pycnonuclear reactions.



classical *M*-*R* relation if *P* only from ideal, degen. e^-

White dwarfs

REMINDER: (HUSK)

Crystallization – electrostatic interactions (non-ideal EOS effects)

- so far any interaction between ions were neglected (= ideal gas)
- not valid for high ρ and low T.

-if thermal kinetic energy kT becomes similar to electrostatic (potential) binding energy (Coulomb energy) per ion with charge Ze, ions tend to form a **rigid lattice** $\rightarrow$ minimizes their total energy.

Def.: coupling parameter $\Gamma_c = \frac{\text{potential (Coulomb) binding energy}}{\text{(thermal) kinetic energy}}$

$$\Gamma_c = \frac{(Ze)^2}{r_0 kT} \simeq 2.7 \times 10^{-3} \frac{Z^2\, n_{ion}^{1/3}}{T}$$

$\Gamma_c << 1$ ... ions have M-B distribution

$\Gamma_c >> 1$ ... ions try to form a crystal that has a lower energy

$-Ze$ ... ion charge

r_0 ... mean ion separation

$n_{ion} \frac{4\pi r_0^3}{3} = 1$

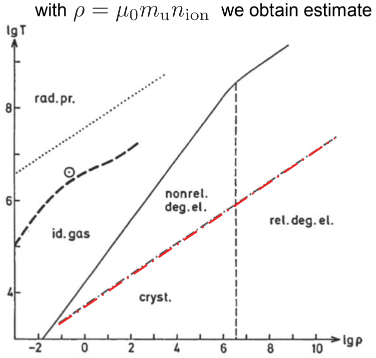
REMINDER:
(HUSK)

White dwarfs

Crystallization – electrostatic interactions (non-ideal EOS effects)

Critical value for transition (Shapiro & Teukolsky 1983): $\Gamma_c \simeq 100$

with $\rho = \mu_0 m_u n_{ion}$ we obtain estimate for critical (melting) temperature T_m:



$$T_m \approx \frac{Z^2 e^2}{\Gamma_c k} \left(\frac{4\pi\varrho}{3\mu_0 m_u} \right)^{1/3}$$

$$= 2.3 \times 10^3 Z^2 \mu_0^{-1/3} \varrho^{1/3}$$

Such conditions are found in cooling white dwarfs

White dwarfs

Crystallization – electrostatic interactions (non-ideal EOS effects)

- **electrostatic correction** to EOS necessary, because positive charges are not uniformly distributed in the *gas* but are concentrated in individual nuclei of positive charge *Ze*.
- the repelling electrons are, on average, further apart then the mean distance between nuclei and electrons, so repulsion is weaker than attraction.

→ this decreases the energy and pressure of the ambient electrons!

- for cold plasma, i.e. $T \to 0$, the ions are located in a lattice that maximizes the inter-ion separation and the *electrons are assumed to be uniformly distributed*.
- for WD densities ($\rho \approx 10^6$ gcm^{-3}) the " *Wigner-Seitz approximation*" can be used, in which we consider spherical cells of the lattice of volume

$$\frac{4\pi r_0^3}{3} = \frac{1}{n_{ion}} = \frac{Z}{n_e}$$

i.e. the gas is imagined to be divided into *neutral spheres* of radius r_0 about each nucleus, which contain the *Z electrons* closest to the nucleus.

White dwarfs

Crystallization – electrostatic interactions (non-ideal EOS effects)

- total electrostatic energy of any one sphere is the sum of potential energies due to electron-electron (e-e) and electron-ion (e-i) interactions.
- to assemble a uniform sphere of Z e^- requires energy

$$E_{e-e} = \int_0^{r_0} \frac{q}{r}\,dq\,, \qquad \text{where} \qquad q = -Ze\frac{r^3}{r_0^3}\,,$$

$$E_{e-e} = \frac{3}{5}\frac{Z^2e^2}{r_0}$$

- to assemble the electron sphere about the central nucleus of charge *Ze* requires energy

$$E_{e-i} = Ze \int_0^{r_0} \frac{dq}{r} = -\frac{3}{2}\frac{Z^2e^2}{r_0}\,.$$

Thus the total Coulomb energy E_c of the cell is

$$E_c = E_{e-e} + E_{e-i} = -\frac{9}{10}\frac{Z^2e^2}{r_0}\,.$$

White dwarfs

Crystallization – electrostatic interactions (non-ideal EOS effects)

The electrostatic energy per electron therefore is

$$\rho = n_{ion} A m_u = n_e A m_u / Z$$

$$\frac{E_c}{Z} = -\frac{9}{10}\frac{Ze^2}{r_0} = -\frac{9}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{1/3} \simeq -2\frac{Z}{A^{1/3}}\rho_6^{1/3}\,\text{keV}\,.$$

$r_0 = (3Z/4\pi n_e)^{1/3}$; $\rho_6 = \rho \cdot 10^{-6}\,\text{g}\,\text{cm}^{-3}$; atomic (integer) number of ion

Note (E.E. Salpeter 1961) used notation:

$$r_0 = Z^{1/3} r_e a_0\,,$$

where $r_e a_0$ is the radius of a sphere that contains on average one electron, with a_0 being the Bohr radius $a_0 = \hbar/\alpha m_e c$ $(\alpha \simeq 1/137)$.

White dwarfs

Crystallization – electrostatic interactions (non-ideal EOS effects)

Zero-point ($T \to 0$) energy E_{ZP} of ions (quantum-mechanical effect)

Even @ $T \to 0$ ions are not at rest in the lattice, but vibrate around their position with the vibrate frequency

$$\omega_E = \frac{1}{\sqrt{3}}\Omega_P$$

ion plasma frequency

$$\Omega_P = \left(\frac{4\pi n_{ion} Z^2 e^2}{m_{ion}}\right)^{1/2}$$

such that their zero-point energy per ion is $E_{ZP} = 3\hbar\omega_E/2.$

With $\rho = n_{ion} A m_u$ the zero-energy per electron is

$$\frac{E_{ZP}}{Z} = \frac{3}{2}\left(\frac{4\pi}{3}\right)^{1/2}\frac{\hbar e}{A m_u}\rho^{1/2} \simeq \frac{0.6}{A}\rho_6^{1/2}\,\mathrm{keV}\,.$$

White dwarfs

Crystallization – electrostatic interactions (non-ideal EOS effects)

Coulomb energy per electron:

$$\frac{E_c}{Z} \simeq -2\frac{Z}{A^{1/3}}\rho_6^{1/3}\,\mathrm{keV}\,,$$

Zero-point energy per electron:

$$\frac{E_{ZP}}{Z} \simeq \frac{0.6}{A}\rho_6^{1/2}\,\mathrm{keV}\,.$$

For ^{12}C (Z=6, A=12) and $\rho = 10^6$ gcm^{-3}:

$$\frac{E_c}{Z} \simeq -5.2\,\mathrm{keV}\,, \qquad \frac{E_{ZP}}{Z} \simeq 0.05\,\mathrm{keV} << -\frac{E_c}{Z}\,.$$

The **ratio**

$$\frac{-E_c}{E_{ZP}} \propto Z A^{2/3}\rho^{1/6}$$

varies only little with ρ and increases towards heavier elements.
Therefore cold WD ("black dwarfs") are crystallized.

Ions form regular lattice which minimizes the energy; they perform low-energy oscillations around their average position, where they are kept by mutual repulsive forces.

White dwarfs

Crystallization – electrostatic interactions (non-ideal EOS effects)

Therefore cold WD ("black dwarfs") are crystallized. The ions, located in the lattice, perform low-energy oscillations around their average positions, where they are kept by mutual repulsive forces.

The total energy per electron is therefore:

$$Z^{-1}E = Z^{-1}(E_0 + E_c + E_{ZP}) \simeq Z^{-1}(E_0 + E_c) << Z^{-1}E_0,$$

E_0 is the (total) mean energy of e^- for an *ideal* Fermi gas as considered in Chandrasekhar's (*idealized*) theory

How important is the (non-ideal) effect of E_c on the EOS ?

from 1st law of Thermodynamics (TD) we have (only e^- considered):

$$\mathrm{d}(E/Z) = -P\mathrm{d}\left(\frac{1}{n_e}\right) + T\mathrm{d}s + (\text{contributions from chemical potentials})\,,$$

$$\longrightarrow \quad P = -\left.\frac{\partial(E/Z)}{\partial\left(\frac{1}{n_e}\right)}\right|_{s,\{n_i\}} = n_e^2 \left.\frac{\partial(E/Z)}{\partial n_e}\right|_{s,\{n_i\}}.$$

REMINDER: (HUSK)

The equation of state of stellar matter

A self-consistent approximate approach

Idea: find a single expression for the EOS from which all thermodynamic quantities e.g., ρ, U, c_p, δ, etc, are consistently derived for given P, T and X_i

Ansatz:
use TD potential of free energy $F(T,V,\{N_i\}) = U - TS$ and find reaction equilibrium by selecting those $\{N_i\}$ that **minimizes *F*** (maximizes entropy S) for given T,V, subject to condition that total numbers of free e^- and any nucleus are constant.

From minimized free energy $F(T,V,\{N_i\})$ all TD quantities can be derived, e.g.

$$P = -\left(\frac{\partial F}{\partial V}\right)_T\,, \quad S = -\left(\frac{\partial F}{\partial T}\right)_V\,, \quad U = -T^2\left(\frac{\partial}{\partial T}\frac{F}{T}\right)_V\,.$$

$$c_p = -\frac{T}{\rho}\left(\frac{\partial^2 F}{\partial T^2}\right)_V\,.$$

White dwarfs

Crystallization – electrostatic interactions
(non-ideal EOS effects)

$$P \simeq n_e^2 \frac{\partial(E_0/Z)}{\partial n_e}\bigg|_{s,\{n_i\}} + n_e^2 \frac{\partial(E_c/Z)}{\partial n_e}\bigg|_{s,\{n_i\}} = P_0 + P_c < P_0 \quad ; \; P_c = -\frac{3}{10}\left(\frac{4\pi}{3}\right)^{1/3} Z^{2/3} e^2 n_e^{4/3} \;.$$

P_c is negative, i.e. reducing the total pressure (and total energy E)

ideal gas with

$$x \to 0 : \quad P_0 = P_e = \frac{1}{20}\left(\frac{3}{\pi}\right)^{1/3} \frac{h^2}{m_e} n_e^{5/3} \quad \to \quad \frac{P}{P_0} = 1 - \frac{Z^{2/3}}{2^{1/3}\pi a_0 n_e^{1/3}} \;.$$

$$x \to \infty : \quad P_0 = P_e = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8} n_e^{4/3} \quad \to \quad \frac{P}{P_0} = 1 - \frac{2^{5/3}}{5}\left(\frac{3}{\pi}\right)^{1/3} \alpha Z^{2/3} \;.$$

Table 37.2. Values of P/P_0, where P includes the Coulomb interaction and P_0 is for an ideal Fermi gas. x is the relativity parameter; ϱ is in g cm^{-3}. (After Salpeter, 1961)

x	$\varrho \cdot 2/\mu_e$	P/P_0 ($Z=2$)	P/P_0 ($Z=6$)	P/P_0 ($Z=26$)
0.05	2.44×10^2	0.760	0.564	−0.063
0.1	1.95×10^3	0.880	0.782	0.467
1	1.95×10^6	0.988	0.975	0.933

approximation breaks down

P reduction increases with Z and decreasing $\rho \to P_c$ most important at small M ($\to$ reduction of R).

The equation of state of stellar matter

Crystallization

- *so far any interaction between ions were neglected (= ideal gas)*
- *not valid for high ρ and low T.*

-if thermal kinetic energy kT becomes similar to electrostatic (potential) binding energy (Coulomb energy) ions tend to form a rigid lattice -> minimizes their total energy

Def.: coupling parameter $\Gamma_c = \dfrac{\text{potential (Coulomb) binding energy}}{\text{(thermal) kinetic energy}}$

$$\Gamma_c = \frac{(Ze)^2}{r_{ion} kT} \simeq 2.7 \times 10^{-3} \frac{Z^2 n_{ion}^{1/3}}{T}$$

$\Gamma_c << 1$ *... ions have B-distribution*

$\Gamma_c >> 1$ *... ions try to form a crystal that has a lower energy*

$-Ze$ *... ion charge*

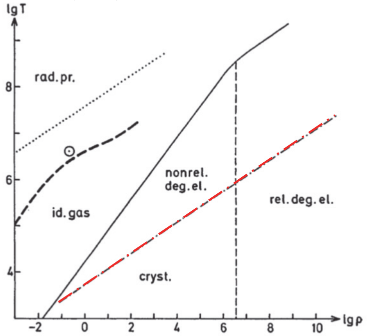
r_{ion} *... mean separation between ions*

The equation of state of stellar matter

Crystallization

Critical value for transition (Shapiro & Teukolsky 1983): $\Gamma_c \simeq 170$.

with $\rho = \mu_0 m_u n_{ion}$ *we obtain estimate for critical (melting) temperature* T_m:



$$T_m \approx \frac{Z^2 e^2}{\Gamma_c k}\left(\frac{4\pi\varrho}{3\mu_0 m_u}\right)^{1/3}$$

$$= 2.3 \times 10^3 Z^2 \mu_0^{-1/3} \varrho^{1/3}$$

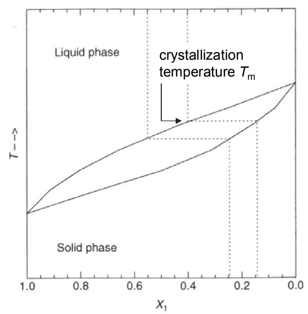
Such conditions are found in cooling white dwarfs

White dwarfs

Chemical or *phase separation*
(non-ideal EOS effect)

Apart from reducing the pressure, crystallisation changes also the chemical structure, as a result that different elements cannot coexist in arbitrary amounts in the solid (crystallized) phase.

Phase diagram



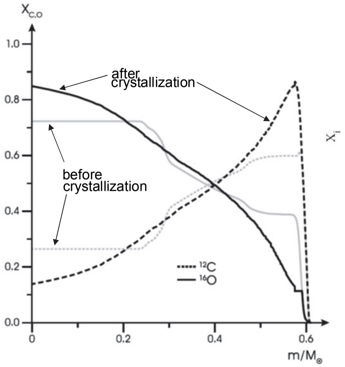
The abundance ratio X_c/X_o of a CO binary mixture in the liquid (gaseous) phase cannot be maintained in the solid (crystallized) phase, because the two elements are not fully miscible in the solid phase.

Assume mass fractions X_1 (lighter element C) & X_2=1-X_1; with X_1=0.4 at the start of crystallization (T_m); in the solid phase X_1=0.14 →X_1 @ crystallization boundary (CB) is necessarily increased, i.e. the excess amount of C will flow up the liquid phase, reducing @ CB the mean molecular weight (density inversion) and convection sets in, enhancing X_1 everywhere in liquid phase, to e.g. X_1=0.55, thereby decreasing X_2 @ CB; this layer will subsequently crystallize at a lower temperature T_m & the cycle is repeated again, now with, e.g., X_1=0.25 @ the crystallized WD centre, until the whole degenerated core is crystallized.

The final profiles of X_1 & X_2 will no longer be homogeneous, but X_2 will decrease from the centre & X_1 will increase.

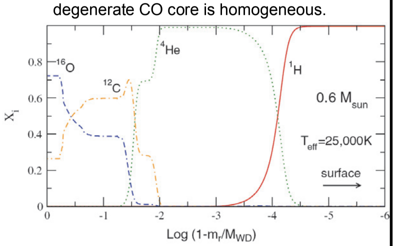
White dwarfs

Chemical or *phase separation* (non-ideal EOS effect)



0.609 M⊙ WD (initial mass 2M⊙)

Before crystallization, i.e. after the AGB phase (He-shell burning) the chemical profile of the degenerate CO core is homogeneous.



After crystallization the final profiles of X_c & X_o will no longer be homogeneous, but X_o will decrease from the centre & X_c will increase.

This modification in the chemical structure will affect the mechanical structure of the WD and therefore also the EOS (**non-ideal effect**);

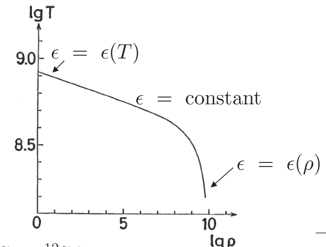
it also releases gravitational energy $\varepsilon_g > 0$!

White dwarfs

Pycnonuclear reactions @ high densities

- these are nuclear reactions (not thermonuclear!), which depend mainly on ρ (and less on T) and can occur at $T \to 0$ (making use of 'tunnel effect'):

- strong screening: $E_D = \frac{Z_1 Z_2 e^2}{r_D}$ $\quad \frac{E_D}{kT} \gg 1 \rightarrow \frac{E_D}{kT} \propto \frac{\rho^{1/3}}{T}$ $\longrightarrow$ screening factor $f \sim \rho^\lambda$ depends strongly on ρ ($\lambda > 1$)



$^{12}C + {}^{12}C$ burning

Reactions set in rather abruptly at a certain density limit ρ_{pyc} , and use all the fuel within a short time (~ 10^5 y), once $\rho > \rho_{pyc}$.

element	ρ_{pyc} gcm^{-3}
^1H	10^6
^4He	10^9
^{12}C	10^{10}

WD central densities: $\rho \approx 10^6 - 10^9$ gcm^{-3}

$\longrightarrow$ only relevant in extreme WD cases.

White dwarfs

Inverse β decay (neutronization, e- capture) @ high densities

$$(Z,A) + e^- \rightarrow (Z-1,A) + \nu_e$$

High-energy e⁻ can combine with protons to form neutrons if total e⁻ energy is:

$$E_{tot} > E^* = c^2(m_n - m_p) \; .$$

At relatively low ρ the neutron will decay within ~11 min to produce proton-e⁻ pair with ejected e⁻ having energy $E^\star_{kin} = E^\star - m_e c^2$.

However, for **complete e⁻ degeneracy** Fermi energy $E_F = m_e c^2[(1+x^2)^{1/2} - 1]$ could $> E^\star_{kin}$ and released e⁻ have not enough energy to find empty cell in phase space, i.e.

$\longrightarrow$ neutron can not decay, or in other words,

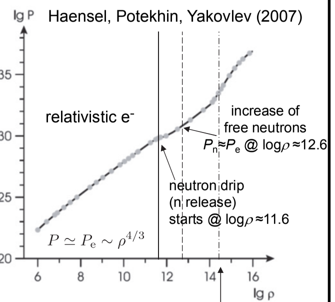
$\longrightarrow$ **Fermi sea** stabilizes neutrons if $E^\star_{kin} \simeq E_F \propto \rho^{2/3} \longrightarrow \rho_{crit}$

i.e. for $\rho > \rho_{crit}$ $\longrightarrow$ neutronization (neutron gas).

$\rho_{crit} \simeq 1.2 \times 10^7$ gcm^{-3} for ^1H
$\rho_{crit} \simeq 1.4 \times 10^{11}$ gcm^{-3} for ^4He
$\rho_{crit} \simeq 3.9 \times 10^{10}$ gcm^{-3} for ^{12}C

$\rho_n := \rho_{crit} >> \rho_{pyc}$

Haensel, Potekhin, Yakovlev (2007)



nuclei dissolved: degen. n-gas & free p & e⁻

only for heavy elements $\rho_n << \rho_{pyc}$

e.g., $^{56}Fe \rightarrow {}^{56}Mn \rightarrow {}^{56}Cr$:
$\rho_n = 1.14 \times 10^9$ gcm^{-3} $< \rho_{pyc}$

White dwarfs

Nuclear equilibrium @ high densities

- in the nuclear reaction chapter we discussed that ^{56}Fe is the most stable element, beyond which (in A) no further thermonuclear reactions can take place.

- the counteraction of attracting nuclear and repelling Coulomb forces gives a maximum binding of the nucleons at ^{56}Fe; therefore **^{56}Fe** will be the **equilibrium composition** for **small densities ρ < 8x10^6 gcm^{-3}**.

- the equilibrium composition is that composition, for which the total energy is a minimum, which may be obtained only after a very long time. (Basically one imposes only the baryon number per volume and ask for the corresponding equilibrium composition.)

- with increasing ρ the balance between attractive and repelling forces in isolated nuclei (Z,A) is shifted to heavier and neutron-enriched nuclei, because replacing a proton by a neutron decreases the repulsive Coulomb force inside the nucleus, and resulting β decay is inhibited by Fermi sea ($E_F \sim \rho^{2/3}$).

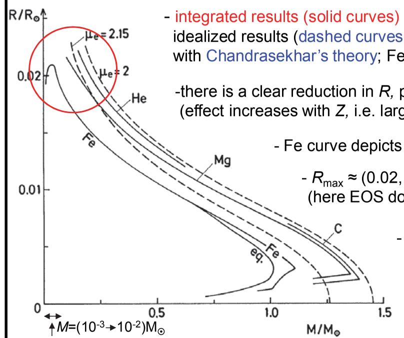
- sequence of equilibrium nuclei (ρ_{max}): $^{56}Fe(8\times10^6)$, $^{62}Ni(2.8\times10^8)$, $^{64}Ni(1.3\times10^9)$, $\ldots$, $^{120}Sr(3.6\times10^{11})$, $^{122}Sr(3.8\times10^{11})$, $^{118}Kr(4.4\times10^{11})$.

- for $\rho > 4\times10^{11}$gcm^{-3}, energetically more favourable to have free instead of bound neutrons $\longrightarrow$ *neutron drip*. (binding energy of last neutron in nucleus ~ 0).

White dwarfs

Effects of using a more realistic EOS: the M-R relation (Hamada & Salpeter 1961)

- with a more 'complete' EOS (still for low *T* only), the mechanical equations can be integrated outwards for various values for the central pressure, which leads to a pair of values for *M* & *R*.



M=($10^{-3}\rightarrow10^{-2}$)M⊙
EOS dominated by Coulomb interactions & **T**homas-**F**ermi correction (inhomog. e⁻ distr.)

- integrated results (solid curves) are compared with idealized results (dashed curves, μ_e=2 & 2.15 (Fe)) obtained with Chandrasekhar's theory; Fe: $M_{Ch}\approx1.25M_\odot$.

- there is a clear reduction in *R*, particularly @ small *M*, because of E_c ($P\downarrow$)! (effect increases with *Z*, i.e. largest for Fe).

- Fe curve depicts R_{max} @ small *M* and decreases for $M\rightarrow0$.

- $R_{max}\approx(0.02, 0.05, 0.12)R_\odot$ for Fe, He & H, respectively; (here EOS dominated by Coulomb interactions & T-F corr.)

- R_{max} defines natural transition between **WD** ($dR/dM < 0$) and **planets** ($dR/dM > 0$).

-note: **Jupiter** with $R\approx0.1R_\odot$ & $M\approx10^{-3}M_\odot$ is close to $R_{max}\approx0.12M_\odot$, suggesting that Jupiter **consists mostly of H**.

White dwarfs

Effects of using a more realistic EOS: the M-R relation (Hamada & Salpeter 1961)

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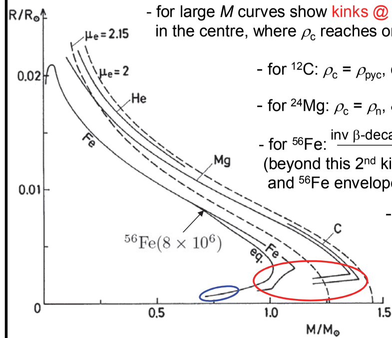


- for large *M* curves show kinks @ mass limit, as a result of a *phase transition* in the centre, where ρ_c reaches one of limits (ρ_{pyc}, ρ_n, ρ_{max}), discussed before:

  - for ^{12}C: $\rho_c=\rho_{pyc}$, converting $^{12}C\rightarrow{}^{24}Mg\xrightarrow{\text{inv }\beta\text{-decay}}{}^{24}Ne$.

  - for ^{24}Mg: $\rho_c=\rho_n$, & $\xrightarrow{\text{inv }\beta\text{-decay}}{}^{24}Ne$.

  - for ^{56}Fe: $\xrightarrow{\text{inv }\beta\text{-decay}}{}^{56}Cr$ @ M_{max} , and $\xrightarrow{\text{inv }\beta\text{-decay}}{}^{56}Ti$ @ 2nd kink; (beyond this 2nd kink models consist of ^{56}Tl cores, ^{56}Cr shells, and ^{56}Fe envelopes).

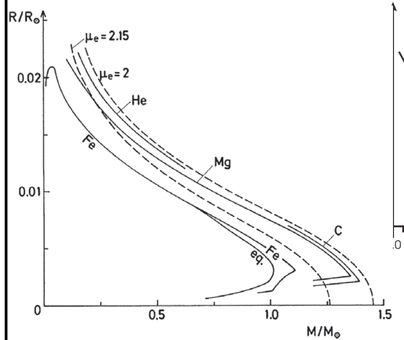
- for *equilibrium composition*: coincides with Fe solution for $\rho<8\times10^6$gcm^{-3} and is below all other curves, because of largest average μ_e!

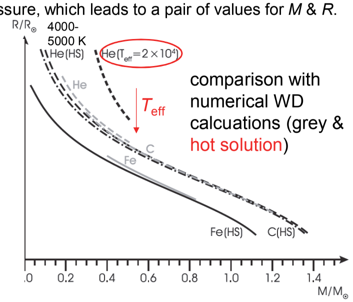
at $M_{max}\approx1M_\odot$, $\rho_c\approx2\times10^9$gcm^{-3}, with ^{66}Ni nuclei; near the 'end' of the curve, ^{120}Kr is reached and **first neutrons are freed**.
From here it follows equilibrium configurations which leads to **neutron stars**.

White dwarfs

Effects of using a more realistic EOS: the M-R relation (Hamada & Salpeter 1961)

- with a more 'complete' EOS (still for low *T* only), the mechanical equations can be integrated outwards for various values for the central pressure, which leads to a pair of values for *M* & *R*.





comparison with numerical WD calcuations (grey & hot solution)

Hamada & Salpeter (HS) solutions in good agreement with cold-*T* solutions.

- hotter WD have (initially) larger R, but approach cold configuration during cooling process.