The onset of star formation

The Jeans criterion

- Consider infinite, homogeneous, gas cloud at rest ($\boldsymbol{v}$=0) with T_0=const & ρ_0=const.
- For symmetry reasons (Euler equation), gravitational potential Φ must also be constant, but Poisson's equation would demand $\rho_0 = 0$ ⟶ not well defined equilibrium (Jeans swindle).
- Momentum equation (Euler equation)

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\varrho}\nabla P - \nabla \Phi$$

- continuity equation $\quad \dfrac{\partial \varrho}{\partial t} + \boldsymbol{v}\nabla\varrho + \varrho\nabla \cdot \boldsymbol{v} = 0$
- Poisson's equation $\quad \nabla^2 \Phi = 4\pi G \varrho$
- EOS (ideal gas) $\quad P = \dfrac{\Re}{\mu}\varrho T = v_{\mathrm{s}}^2 \varrho \;.$ ← (thermal) sound speed

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- For symmetry reasons (Euler equation), gravitational potential Φ must also be constant, but Poisson's equation would demand $\rho_0 = 0$ ⟶ not well defined equilibrium (**Jeans swindle**).
- Momentum equation (Euler equation)

$\boldsymbol{v}$=0: P=const ⟶ Φ = const

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\varrho}\nabla P - \nabla \Phi$$

- continuity equation $\quad \dfrac{\partial \varrho}{\partial t} + \boldsymbol{v}\nabla\varrho + \varrho\nabla \cdot \boldsymbol{v} = 0$
- Poisson's equation $\quad \nabla^2 \Phi = 4\pi G \varrho$ ⟵ only possible if $\rho = 0$ ⟶ contradicts
- EOS (ideal gas) $\quad P = \dfrac{\Re}{\mu}\varrho T = v_{\mathrm{s}}^2 \varrho \;.$ ← (thermal) sound speed

The onset of star formation

The Jeans criterion

- Perturb equilibrium by setting

$$\varrho = \varrho_0 + \varrho_1\,, \quad P = P_0 + P_1\,, \quad \Phi = \Phi_0 + \Phi_1\,, \quad \boldsymbol{v} = \boldsymbol{v}_1\,,$$

where subscript 1 denotes small perturbations allowing linearization.

- Insert these into previous three conservation equations and linearize perturbations to obtain

$$\frac{\partial \boldsymbol{v}_1}{\partial t} = -\nabla\left(\Phi_1 + v_{\mathrm{s}}^2 \frac{\varrho_1}{\varrho_0}\right),$$

$$\frac{\partial \varrho_1}{\partial t} + \varrho_0 \nabla \cdot \boldsymbol{v}_1 = 0\,,$$

$$\nabla^2 \Phi_1 = 4\pi G \varrho_1\,.$$

The onset of star formation

The Jeans criterion

- Linear, homogeneous system of DEs with constant coefficient can be solved with exponential dependence in time and space, e.g. for a "simple geometry"

$$x_1 \propto \mathrm{e}^{\mathrm{i}(kx+\omega t)}\,,$$

i.e. $\quad \dfrac{\partial}{\partial x} = \mathrm{i}k\,, \quad \dfrac{\partial}{\partial y} = \dfrac{\partial}{\partial z} = 0\,, \quad \dfrac{\partial}{\partial t} = \mathrm{i}\omega\,.$

With $\quad v_{1x} = v_1, v_{1y} = v_{1z} = 0$

$$\omega v_1 + \frac{k v_{\mathrm{s}}^2}{\varrho_0}\varrho_1 + k\Phi_1 = 0\,,$$

$$k\varrho_0 v_1 + \omega\varrho_1 = 0\,,$$

$$4\pi G\varrho_1 + k^2\Phi_1 = 0\,.$$

The onset of star formation
The Jeans criterion

$$\omega v_1 + \frac{k v_s^2}{\varrho_0}\varrho_1 + k\Phi_1 = 0\,,$$
$$k\varrho_0 v_1 + \omega\varrho_1 = 0\,,$$
$$4\pi G\varrho_1 + k^2\Phi_1 = 0\,.$$

This homogeneous set of linear equations has a non-trivial solution only if the determinant

$$\det \begin{vmatrix} \omega & \frac{k v_s^2}{\rho_0} & k \\ k\varrho_0 & \omega & 0 \\ 0 & 4\pi G & k^2 \end{vmatrix} = 0\,.$$

$\longrightarrow$ $\boxed{\omega^2 = k^2 v_s^2 - 4\pi G\varrho_0\,.}$ = Jeans dispersion equation

The onset of star formation
The Jeans criterion

$$\omega^2 = k^2 v_s^2 - 4\pi G\varrho_0\,.$$

For $k^2 > 4\pi G\varrho_0/v_s^2$ is ω real and oscillatory solution has constant amplitude $\longrightarrow$ stable.

If $k^2 < 4\pi G\varrho_0/v_s^2$ ω is imaginary and of the form $\pm i\xi$, i.e. the amplitude of the perturbations, x_1, can grow with time t according to

$$x_1 \propto e^{\mp\xi t} e^{ikx}\,,$$

i.e. the solution (equilibrium) is unstable.

$\longrightarrow$ characteristic wavenumber $k_J^2 := \frac{4\pi G\varrho_0}{v_s^2}\,,$

& characteristic wavelength $\lambda_J := \frac{2\pi}{k_J} = \left(\frac{\pi}{G\varrho_0}\right)^{1/2} v_s\,.$ **Jeans criterium**

The onset of star formation
The Jeans criterion

$$\omega^2 = k^2 v_s^2 - 4\pi G\varrho_0\,.$$

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& characteristic wavelength $\lambda_J := \frac{2\pi}{k_J} = \left(\frac{\pi}{G\varrho_0}\right)^{1/2} v_s\,.$ **Jeans criterium**

For our given choice of perturbation (geometry), instability due to a slight compression of a set of plane-parallel slabs leads to a "faster" increase of gravity than pressure and slaps collapse to thin sheets. During collapse we have (gravity dominates over "pressure" $k^2 v_s^2$)

$$i\omega \approx (G\varrho_0)^{1/2}$$

$\longrightarrow$ characteristic time scale $\tau \approx (G\varrho_0)^{-1/2}$, which corresponds to τ_{ff}.

The onset of star formation
Plane-parallel <u>stratified</u> layer in hydrostatic equilibrium

Well-defined equilibrium is defined for an isothermal plane-parallel layer (disc) stratified according to hydrostatic equilibrium in z direction:

$$\frac{d^2\Phi_0}{dz^2} = 4\pi G\varrho_0\,, \quad \text{and} \quad dP_0/dz = -\varrho_0 d\Phi_0/dz,$$

using EOS $\longrightarrow$ $v_s^2 \frac{d\,\ln\varrho_0}{dz} = -\frac{d\Phi_0}{dz}\,.$

After differentiation and use of Poisson equation $\frac{d^2\ln\varrho_0}{dz^2} = -\frac{4\pi G}{v_s^2}\varrho_0$

with the solution ($\varrho_0 = 0$ for $z = \pm\infty$) $\varrho_0(z) = \frac{\varrho_0(0)}{\cosh^2(z/H)}\,,$

where $H = \left(\frac{\Re T}{2\pi\mu G\varrho_0(0)}\right)^{1/2} = \frac{v_s}{[2\pi G\varrho_0(0)]^{1/2}}\,.$

Investigate stability with perturbation $\varrho_1 \sim f(z)\exp[i(kx+\omega t)]$:

$\longrightarrow$ critical wavenumber: $k_J = \frac{1}{H} = \frac{[2\pi G\varrho_0(0)]^{1/2}}{v_s} = \sqrt{2}\,k_{J,\,Jeans}\,.$ Note: instability if $k < k_J$
(Spitzer 1968)

The onset of star formation
Instability in the spherical case

- Consider **isothermal sphere** of finite *R* and ideal gas embedded in medium with pressure *P**.
- Structure of sphere can be obtained from Lane-Emden equ. for isothermal polytrope; solution is cut @ radius *R* where *P*=*P**. We assume *P**=const. during perturbation.

Internal energy of *isothermal sphere* of mass *M*: $E_\mathrm{i} = c_\mathrm{v} M T\ ;$

Gravitational energy : $E_\mathrm{g} = -\Theta \frac{GM^2}{R}\ ;$

order of unity; obtained from Lane-Emden equ.: $\Theta = 3/(5-n)$.

For ideal monoatomic gas (ζ=2) the virial theorem for non-vanishing surface pressure P_0 gives

$$P_0 = \frac{c_v M T}{2\pi R^3} - \frac{\Theta G M^2}{4\pi R^4}\ .$$

The onset of star formation
Instability in the spherical case

$$P_0 = \frac{c_v M T}{2\pi R^3} - \frac{\Theta G M^2}{4\pi R^4}\ .$$

from internal gas pressure trying to expand sphere; from self-gravity, trying to bring matter to centre

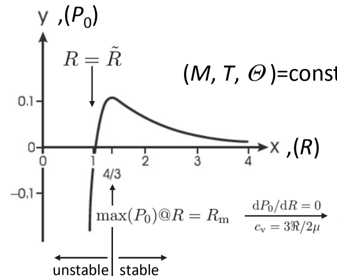
Introduce two scaling factors to render equations dimensionless

$$\widetilde{R} = \frac{\Theta G M}{2 c_v T}\,, \qquad \widetilde{P} = \frac{c_v M T}{2\pi \widetilde{R}^3}\,, \qquad \longrightarrow \qquad R = x\widetilde{R}, \qquad P_0 = y\widetilde{P}.$$

$$y = \frac{1}{x^3}\left(1 - \frac{1}{x}\right).$$

The onset of star formation
Instability in the spherical case

$$P_0 = \frac{c_v M T}{2\pi R^3} - \frac{\Theta G M^2}{4\pi R^4}\ . \quad \xrightarrow[P_0 = y\widetilde{P}]{R = x\widetilde{R}} \quad y = \frac{1}{x^3}\left(1 - \frac{1}{x}\right).$$



(*M*, *T*, *Θ*)=constant

$$\max(P_0)@R = R_\mathrm{m} \quad \xrightarrow[c_\mathrm{v} = 3\Re/2\mu]{\mathrm{d}P_0/\mathrm{d}R = 0} \quad R_\mathrm{m} = \frac{4\Theta}{9}\frac{G\mu M}{\Re T} = \frac{4}{3}\widetilde{R}\,.$$

With $M = 4\pi R_\mathrm{m}^3 \bar{\varrho}/3$ (mean density of sphere) $\longrightarrow R_\mathrm{m}^2 = \frac{27}{16\pi\Theta}\frac{\Re T}{G\mu\bar{\varrho}} \longleftrightarrow \boxed{\lambda_\mathrm{J}^2 = \frac{\Re T \pi}{G\mu\bar{\varrho}}}.$

The onset of star formation
Instability in the spherical case

$$R_\mathrm{m}^2 = \frac{27}{16\pi\Theta}\frac{\Re T}{G\mu\bar{\varrho}}$$

For given equilibrium state with *R* & surface P_0 there exists critical M_J=Jeans mass, where $R=R_\mathrm{m}$

$$M_\mathrm{J} = \frac{4\pi}{3}\bar{\varrho}R_\mathrm{m}^3 = \frac{27}{16}\left(\frac{3}{\pi}\right)^{1/2}\left(\frac{\Re}{\Theta G}\right)^{3/2}\left(\frac{T}{\mu}\right)^{3/2}\left(\frac{1}{\bar{\varrho}}\right)^{1/2}\ .$$

Rewrite with *Θ*=1

$$M_\mathrm{J} = \frac{27}{16}\left(\frac{3}{\pi}\right)^{1/2}\left(\frac{\Re}{G\mu}\right)^{3/2} T^{3/2}\varrho^{-1/2}$$
$$= 1.1\,M_\odot \left(\frac{T}{10\mathrm{K}}\right)^{3/2}\left(\frac{\varrho}{10^{-19}\mathrm{g\ cm^{-3}}}\right)^{-1/2}\left(\frac{\mu}{2.3}\right)^{-3/2}\ .$$

The onset of star formation
Instability in the spherical case

$$M_J = \frac{27}{16}\left(\frac{3}{\pi}\right)^{1/2}\left(\frac{\Re}{G\mu}\right)^{3/2} T^{3/2}\varrho^{-1/2}$$
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typical for star-forming clumps; H in molecular form, He = neutral

For molecular cloud as a whole: $T \approx 100$K and $\rho \approx 10^{-24}$ gcm^{-3} ⟶ $M_J \approx 10^5\ M_\odot$.

Time scale for growth of instability $\tau \approx (G\varrho)^{-1/2}$, the free-fall time.

For $\rho \approx 10^{-19}$ gcm^{-3} ⟶ $\tau \simeq 10^5$ y.

During collapse $\rho\uparrow$ ⟶ $\tau\downarrow$.

The onset of star formation
Fragmentation

- To actually form stars from clouds of ~$10^5 M_\odot$, fragmentation into smaller clumps is required.
- Molecular clouds are highly turbulent with **supersonic motions**, depositing E_{kin} into cloud, stabilizing it against gravitational collapse.
- Same **shock waves** on smaller scales result in local compression ⟶ gravoturbulent cloud fragmentation leads to overdense gas filaments and clumps ⟶ $M > M_J$ ⟶ collapse.

  For isothermal collapse: M_J decreases as $\rho^{-1/2}$

  For adiabatic collapse: M_J increases as $\rho^{1/2}$, because $T \sim P^{2/5} \sim \rho^{2/3}$ & $M_J \sim T^{3/2}\rho^{-1/2}$;
  because $\tau_{adj} << \tau_{ff}$ one can assume for (initial, optically thin gas) collapse to be isothermal.

- Decreasing M_J (with increasing ρ) leads eventually to fragmentation, e.g. if M_J has dropped below, e.g. ½ original M_{cloud}, clump can split into two independently collapsing fragments. Fragmentation continues as long as gas remains isothermal.

- Fragmentation stops when gas becomes opaque (optically thick) and heat gained from gravothermal contraction can no longer be radiated away, i.e. roughly if $\tau_{adj} \approx \tau_{ff}$.

The onset of star formation
Instability in the spherical case

Time scale for growth of instability $\tau \approx (G\varrho)^{-1/2}$, the free-fall time.

For $\rho \approx 10^{-19}$ gcm^{-3} ⟶ $\tau \simeq 10^5$ y .

During collapse $\rho\uparrow$ ⟶ $\tau\downarrow$.

Time scale τ >> thermal adjustment time τ_{adj} (for optically thin cloud):

For neutral H cloud, Spitzer (1968) estimates heat loss $\Lambda \approx$ 1 erg g^{-1}s^{-1} ;

⟶ $\tau_{adj} \simeq c_v T/\Lambda \simeq 10\,\mathrm{y}$ for T=10 K.

$\tau_{adj} << \tau$ ⟶ collapse proceeds in thermal adjustment, i.e. almost isothermal.

cloud optically thin for $\rho < 10^{-14}\mathrm{gcm^{-3}}$

The onset of star formation
Fragmentation

- Fragmentation stops when gas becomes opaque (optically thick) and heat gained from gravothermal contraction can no longer be radiated away, i.e. roughly if $\tau_{adj} \approx \tau_{ff}$;
  ⟶ gas is no longer isothermal and $M_J \neq \rho^{-1/2}$ ⟶ fragmentation stops.

- Rough estimate when $\tau_{adj} \approx \tau_{ff}$ according to Rees (1976):

  free-fall time of a fragment: $\tau_{ff} \approx (G\rho)^{-1/2}$

  total energy radiated away during collapse: $E_g \approx GM^2/R$

  ⟶ radiated energy rate A to keep T of fragment (M,R) constant is

$$A \approx \frac{GM^2}{R}(G\varrho)^{1/2} = \left(\frac{3}{4\pi}\right)^{1/2}\frac{G^{3/2}M^{5/2}}{R^{5/2}} .$$

  Fragment with T can not radiate more than a black body (BB) of temperature T:

  $B = 4\pi f\sigma T^4 R^2$, $\sigma = 2\pi^5 k^4/(15c^2h^3)$ is the Stefan–Boltzmann constant,

  $f < 1$ accounts for radiation less then that of a BB.

The onset of star formation
Fragmentation

Fragment with *T* can not radiate more than a black body (BB) of temperature *T*:

$$B = 4\pi f \sigma T^4 R^2 \,, \qquad \sigma = 2\pi^5 k^4/(15 c^2 h^3) \text{ is the Stefan–Boltzmann constant,}$$

$f < 1$ accounts for radiation less then that of a BB.

B >> *A* for isothermal collapse.
B ≈ *A* indicates transition from isothermal to adiabatic collapse, i.e. if

$$M^5 = \frac{64\pi^3}{3} \frac{\sigma^2 f^2 T^8 R^9}{G^3} \,.$$

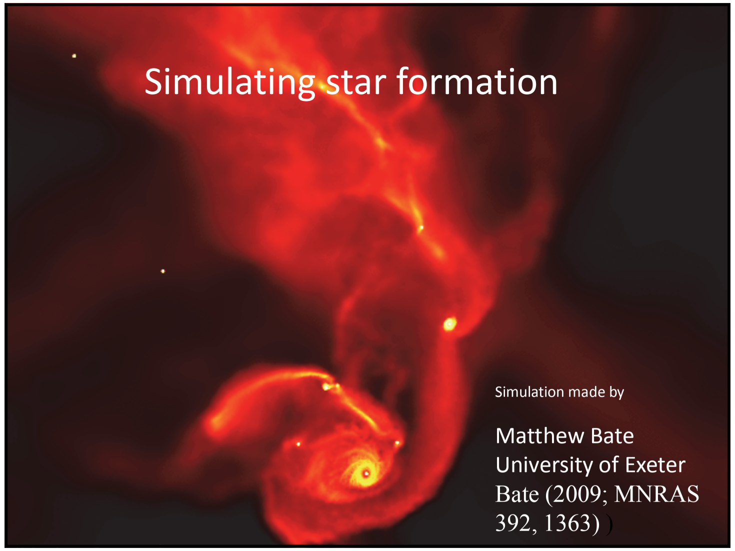
With $M = M_J$ (fragmentation has reached its limit), and $R = \left(\frac{3}{4\pi}\right)^{1/3} \frac{M_J^{1/3}}{\varrho^{1/3}}$,

using $M_J \propto T^{3/2}/\rho^{1/2}$ to eliminate ρ →

$$M_J = \frac{81}{64}\left(\frac{3}{\pi}\right)^{3/4} \frac{1}{(\sigma G^3)^{1/2}} \left(\frac{\Re}{\mu}\right)^{9/4} f^{-1/2} T^{1/4}$$

$$= 6.2 \times 10^{30}\text{g}\ f^{-1/2} T^{1/4} = 0.003 M_\odot \frac{T^{1/4}}{f^{1/2}} \,. \qquad (\mu = 1.)$$

includes dimensions of $(k/m_p c^2)^{1/4} \sim T^{-1/4}$



The onset of star formation
Fragmentation

Jeans mass at end of fragmentation process (i.e. fragmentation stops):

$$M_J = 6.2 \times 10^{30}\text{g}\ f^{-1/2} T^{1/4} = 0.003 M_\odot \frac{T^{1/4}}{f^{1/2}} \,.$$

Example: *T*=10K and appreciable deviations from isothermal collapse when *f* = 0.1

M ≈ 0.001 $M_\odot$ = similar to mass of smallest, optically thick, pressure-supported protostellar cores obtained from numerical simulations.

Note: result depends crucially on chemical composition: the larger *Z* the more radiative cooling. Radiative cooling is very inefficient for Population III stars (*Z*=0), proceeding mainly via H_2 molecules, which dissociate (H_2 → H + H) even at *T* ≈ 2000 K:

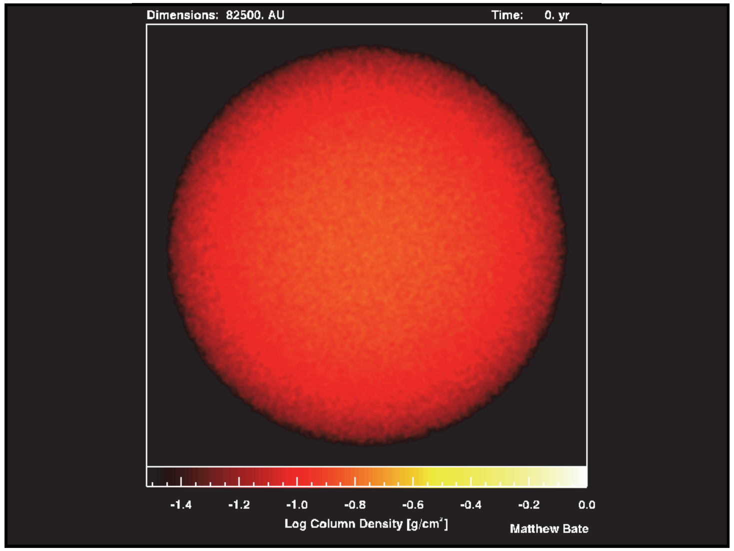
→ smallest condensations in a primordial (Pop. III) collapsing cloud is (Bromm & Larson 2004)

M ≈ 100 $M_\odot$.

The formation of protostars

Bate, M. R., 2009, MNRAS, 392, 1363-1380.
http://www.astro.ex.ac.uk/people/mbate/Cluster/clusterRT.html

Technical Details
The calculations model the collapse and fragmentation of **50 solar mass** molecular clouds that are 0.375 pc or 0.180 pc in diameter (approximately 1.2 and 0.6 light-years, respectively). At the initial temperature of 10 K with a mean molecular weight of 2.38, this results in a thermal Jeans mass of 1 solar mass. The free-fall time of the first type of cloud is 190,000 years and the simulations cover 266,000 years. The free-fall time of the denser cloud is 63,400 years and the simulations cover 89,000 years.
The clouds are given an initial supersonic `turbulent' velocity fields in the same manner as Ostriker, Stone & Gammie (2001). We generate a divergence-free random Gaussian velocity field with a power spectrum P(k) ~k^{-4}, where k is the wave-number. In three-dimensions, this results in a velocity dispersion that varies with distance, lambda, as sigma(lambda) ~ $\text{lambda}^{1/2}$ in agreement with the observed Larson scaling relations for molecular clouds (Larson 1981). This power spectrum is slightly steeper than the Kolmogorov spectrum, P(k)~ $k^{11/3}$. Rather, it matches the amplitude scaling of Burgers supersonic turbulence associated with an ensemble of shocks (but differs from Burgers turbulence in that the initial phases are uncorrelated). The two calculations use the same initial velocity field.
The calculations were performed using a parallel three-dimensional smoothed particle hydrodynamics (SPH) code with 3.5 million particles on the United Kingdom Astrophysical Fluids Facility (UKAFF) and the University of Exeter Supercomputer. They each took approximately 40000 CPU hours running on 8 to 16 processors. The SPH code was parallelised using OpenMP by M. Bate. The code uses sink particles (Bate, Bonnell & Price 1995) to model condensed objects (i.e. the stars and brown dwarfs). Sink particles are point masses that accrete bound gas that comes within a specified radius of them. This accretion radius is to set 0.5 AU. Thus, the calculation resolves circumstellar discs with radii down to approximately 1 AU.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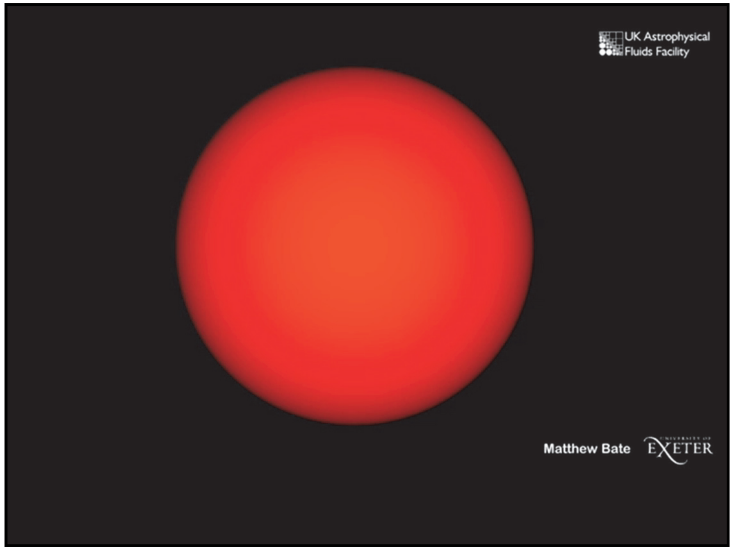
Log Column Density [g/cm²]
Matthew Bate


UK Astrophysical Fluids Facility
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EXETER