The onset of star formation
The Jeans criterion

- Consider infinite, homogeneous, gas cloud at rest \( \mathbf{v} = 0 \) with \( T_0 = \text{const} \) & \( \rho_0 = \text{const} \).

- For symmetry reasons (Euler equation), gravitational potential \( \Phi \) must also be constant, but Poisson’s equation would demand \( \rho_1 = 0 \) \( \rightarrow \) not well defined equilibrium (Jeans swindle).

- Momentum equation (Euler equation)

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P - \nabla \Phi
\]

- Continuity equation

\[
\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \mathbf{v} \cdot \nabla \mathbf{v} = 0
\]

- Poisson’s equation

\[
\nabla^2 \Phi = 4\pi G \rho
\]

- EOS (ideal gas)

\[
P = \frac{\rho}{\mu} T = \rho v^2
\]

(thermal sound speed)

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- Perturb equilibrium by setting

\[
\rho = \rho_0 + \rho_1, \quad P = P_0 + P_1, \quad \Phi = \Phi_0 + \Phi_1, \quad \mathbf{v} = \mathbf{v}_1
\]

where subscript 1 denotes small perturbations allowing linearization.

- Insert these into previous three conservation equations and linearize perturbations to obtain

\[
\begin{align*}
\frac{\partial \mathbf{v}_1}{\partial t} &= -\nabla \left( \Phi_1 + \rho_0 \frac{\rho_1}{\rho_0} \right), \\
\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 &= 0, \\
\nabla^2 \Phi_1 &= 4\pi G \rho_1.
\end{align*}
\]

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The Jeans criterion

- Linear, homogeneous system of DEs with constant coefficient can be solved with exponential dependence in time and space, e.g. for a “simple geometry”

\[
\mathbf{x}_1 \propto e^{(kx + \omega t)},
\]

i.e.

\[
\begin{align*}
\frac{\partial}{\partial x} &= ik, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0, \quad \frac{\partial}{\partial t} = \omega.
\end{align*}
\]

With \( v_{1x} = \mathbf{v}_1, v_{1y} = v_{1z} = 0 \)

\[
\omega v_1 + k v_0^2 \Phi_0 + k \Phi_1 = 0, \quad k v_0 v_1 + \omega \Phi_0 = 0, \quad 4\pi G \rho_0 + k \Phi_0 = 0.
\]
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\[ \omega^2 + \frac{k^2}{\rho_0} \Phi_0 + k \Phi_1 = 0, \]
\[ k \omega_0 + \omega = 0, \]
\[ 4\pi G \rho_0 + k^2 \Phi_1 = 0. \]

This homogeneous set of linear equations has a non-trivial solution only if the determinant

\[ \det \begin{vmatrix} \omega & k^2 & k \\ \frac{k^2}{\rho_0} & \omega & 0 \\ k \Phi_0 & 0 & 4\pi G \end{vmatrix} = 0. \]

\[ \omega^2 = k^2 \sqrt{\rho_0} - 4\pi G \rho_0. \]

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\[ \omega^2 = k^2 \sqrt{\rho_0} - 4\pi G \rho_0. \]

For \( k^2 > 4\pi G \rho_0/\sqrt{\rho_0} \) the oscillatory solution has constant amplitude \( \rightarrow \) stable.

If \( k^2 < 4\pi G \rho_0/\sqrt{\rho_0} \), the amplitude of the perturbations \( \Phi \) grows with time according to

\[ \Phi_1 \propto e^{\frac{k^2}{2} \sqrt{\rho_0} t}, \]

i.e. the solution (equilibrium) is unstable.

\[ \omega^2 = \frac{k^2}{4} \sqrt{\rho_0} - 4\pi G \rho_0. \]

Investigate stability with perturbation

\[ z_1 \sim f(z) \exp \left( \left( k x - \omega t \right) \right): \]

\[ k \approx \frac{1}{H} \left( \frac{2\pi G \rho_0}{v_c} \right)^{1/2} \approx \sqrt{\frac{2\pi G \rho_0}{v_c}}. \]

Note: instability if \( k < k_j \)
The onset of star formation
Instability in the spherical case

- Consider isothermal sphere of finite R and ideal gas embedded in medium with pressure \( p^* \).
- Structure of sphere can be obtained from Lane-Emden eqs. for isothermal polytrope; solution is cut at radius \( R \) where \( P = p^* \). We assume \( P = \text{const.} \) during perturbation.

Internal energy of isothermal sphere of mass \( M \):
\[
\Sigma u = c_v M T.
\]
Gravitational energy:
\[
E_g = -\frac{\Theta G M^2}{R}.
\]
Order of unity obtained from Lane-Emden eqn.: \( \Theta = 3/(5 - n) \).

For ideal monoatomic \((\zeta = 2)\) the virial theorem for non-vanishing surface pressure \( P_0 \) gives
\[
P_0 = \frac{c_v M T}{2\pi R^3} - \frac{\Theta G M^2}{4\pi R^4}.
\]

The onset of star formation
Instability in the spherical case

\[
P^* = \frac{c_v M T}{2\pi R^3} - \frac{\Theta G M^2}{4\pi R^4},
\]
from internal gas pressure trying to expand sphere

\[
\tilde{P} = \frac{c_v M T}{2\pi \tilde{R}^3}, \quad \tilde{R} = \frac{R}{\tilde{R}}, \quad \tilde{P} = \frac{P}{P^*}.
\]

Introduce two scaling factors to render equations dimensionless
\[
y = \frac{1}{\frac{\Theta}{\tilde{R}}} \left( 1 - \frac{1}{x} \right).
\]

The onset of star formation
Instability in the spherical case

\[
P_0 = \frac{c_v M T}{2\pi R^3} - \frac{\Theta G M^2}{4\pi R^4},
\]
\[
R = \frac{R}{\tilde{R}}, \quad P_0 = \frac{P}{P^*}.
\]

\[
y = \frac{1}{\frac{\Theta}{\tilde{R}}} \left( 1 - \frac{1}{x} \right).
\]
The onset of star formation

Instability in the spherical case

$$M_c = \frac{27}{16} \left( \frac{3}{\pi} \right)^{1/2} \left( \frac{G \rho}{T} \right)^{3/2} \frac{R^{1/2} \rho^{-1/2}}{2.3}$$

$$= 1.1 M_\odot \left( \frac{T}{10^4} \right)^{1/2} \left( \frac{10^{-19} \text{g cm}^{-3}}{\rho} \right)^{-1/2} \left( \frac{2.3}{R} \right)^{3/2}$$

typical for star-forming clumps; H in molecular form, He = neutral

For molecular cloud as a whole: \( T = 100 \text{K} \) and \( \rho = 10^{14} \text{g cm}^{-3} \) → \( M_c = 10^5 M_\odot \).

Time scale for growth of instability

$$\tau \approx (G \rho)^{1/2}, \text{the free-fall time.}$$

For \( \rho = 10^{19} \text{g cm}^{-3} \) → \( \tau \approx 10^5 \text{yr} \).

During collapse \( \rho \tau \) → \( R \).

Time scale \( \tau \gg \) thermal adjustment time \( \tau_{\text{adj}} \) (for optically thin cloud):

For neutral H cloud, Spitzer (1968) estimates heat loss \( J = 1 \text{ erg g}^{-1} \text{yr} \):

$$\tau_{\text{adj}} \approx \frac{c_s T}{A} \approx 10^4 \text{yr} \text{ for } T = 10^4 \text{K}.$$  

\( \tau_{\text{adj}} \ll \tau \) → collapse proceeds in thermal adjustment, i.e. almost isothermal.

cloud optically thin for \( \rho < 10^{-14} \text{g cm}^{-3} \)

The onset of star formation

Fragmentation

- To actually form stars from clouds of \( \sim 10^5 M_\odot \), fragmentation into smaller clumps is required.

- Molecular clouds are highly turbulent with supersonic motions, depositing \( E_p \) into cloud, stabilizing it against gravitational collapse.

- Same shock waves on smaller scales result in local compression → gravoturbulent cloud fragmentation leads to overdense gas filaments and clumps → \( M > M_c \rightarrow \) collapse.

For isothermal collapse: \( M_c \) decreases as \( \rho^{-1/2} \)

For adiabatic collapse: \( M_c \) increases as \( \rho^{1/4} \), because \( T \sim \rho^{2/5} \) & \( M_c \sim T^{3/2} \rho^{1/2} \); however, \( \tau_{\text{adj}} \ll \tau \), one can assume for initial, optically thin gas collapse to be isothermal.

- Decreasing \( M_c \) (with increasing \( \rho \)) leads eventually to fragmentation, e.g. if \( M_c \) has dropped below, e.g. \( M_c \) original \( M_\odot \) cloud can split into two independently collapsing fragments. Fragmentation continues as long as gas remains isothermal.

- Fragmentation stops when gas becomes opaque (optically thick) and heat gained from gravothermal contraction can no longer be radiated away, i.e. roughly if \( \tau_{\text{adj}} = \tau \).

The onset of star formation

Fragmentation

- Fragmentation stops when gas becomes opaque (optically thick) and heat gained from gravothermal contraction can no longer be radiated away, i.e. roughly if \( \tau_{\text{adj}} = \tau \);

- gas is now longer isothermal and \( M_c \rho^{1/2} \rightarrow \) fragmentation stops.

- Rough estimate when \( \tau_{\text{adj}} = \tau \) according to Rees (1976):

  free-fall time of a fragment: \( \tau_f = (G \rho)^{1/2} \)

  total energy radiated away during collapse: \( E_r = GM_c \rho R \)

  → radiated energy rate \( A \) to keep \( T \) of fragment \((M, R)\) constant is

  $$A = \frac{GM_c^2}{R} (G \rho)^{1/2} \frac{3}{2}\left(\frac{3}{4\pi}\right)^{1/2} \frac{(G \rho)^{3/2}}{R^{3/2}}.$$  

Fragment with \( T \) can not radiate more than a black body (BB) of temperature \( T \):

$$B = \frac{4\pi}{R^2} T^4 \left(1 + \frac{2}{3} \frac{M}{3M_\odot} \right),$$  

\( J < 1 \) accounts for radiation less than that of a BB.
The onset of star formation

Fragmentation

Fragment with $T$ can not radiate more than a black body (BB) of temperature $T$: 

\[ B = 4\pi f r^4 T^2 \text{,} \]

\[ \sigma = 2\pi h^4 / (15\lambda^3) \text{ is the Stefan-Boltzmann constant,} \]

\[ f < 1 \text{ accounts for radiation less than that of a BB.} \]

\( B \gg A \) for isothermal collapse. \( B = A \) indicates transition from isothermal to adiabatic collapse, i.e. if

\[ M_f = \frac{81}{64} \left( \frac{3}{\pi} \right)^{1/4} \frac{1}{(\sigma G^3)^{1/2}} \left( \frac{B}{\mu} \right)^{1/4} \frac{r^{1/2}}{T^{1/4}} \]

With \( M_i = M_f \) (fragmentation has reached its limit), and

\[ B = \left( \frac{3}{4\pi} \right)^{1/3} \frac{M_i^{5/3}}{\sigma G^3} \]

\[ \left( \mu = 1 \right) \]

Using \( \Delta M = T^2 T/\mu^2 \) to eliminate \( \mu \)

The formation of protostars


http://www.astro.ex.ac.uk/people/mbate/Cluster/clusterRT.html

Technical Details

The calculations model the collapse and fragmentation of 90 solar mass molecular clouds that are 0.375 pc or 0.180 pc in diameter (approximately 1.2 and 0.6 light-years, respectively). At the initial temperature of 30 K with a mean molecular weight of 2.38, this results in a thermal Jeans mass of 1 solar mass. The free-fall time of the first type of clouds is 200,000 years and the simulations cover 266,000 years, the free-fall time of the denser clouds is 63,400 years and the simulations cover 89,000 years. These clouds are given an initial supersonic turbulence velocity fields in the same manner as Ostriker, Stone & Gammie (2003). We generate a divergence-free random Gaussian velocity field with a power spectrum \( P(k) \approx k^{-4} \), where \( k \) is the wave-number. In three dimensions, this results in a velocity dispersion that varies with distance, \( \lambda \), as \( \sigma \approx \lambda^{1/2} \lambda^{1/2} \) in agreement with the observed Larson scaling relations for molecular clouds (Larson 1981). This power spectrum is slightly steeper than the Kolmogorov spectrum, \( P(k) \approx k^{-5} \). Rather, it matches the amplitude scaling of Burgers supersonic turbulence associated with an ensemble of shocks (but differs from Burgers turbulence in that the initial phases are uncorrelated). The two calculations use the same initial velocity field. The simulations were performed using a parallel three-dimensional smoothed particle hydrodynamics (SPH) code with 3.5 million particles on the United Kingdom Astrophysical Fluids Facility (UKAFF) and the University of Exeter Supercomputer. They each took approximately 4000 CPU hours running on 8 to 16 processors. The SPH code was parallelised using OpenMP and MPI. The code uses sink particles (Bate, Bonnell & Price 1995) to model condensed objects (i.e. the stars and brown dwarfs). Sink particles are point masses that accrete bound gas that comes within a specified radius of them. This accretion radius is set to 0.5 AU. Thus, the calculation resolves circumstellar discs with radii down to approximately 3 AU.