Stellar Stability
Perturbation Equations

Time-independent equilibrium model, $r_0(m)$, $P_0(m)$, $T_0(m)$, $l_0(m)$, obeys time-independent (t_{nuc} >> t_{KH} >> t_{hydr}) stellar structure equations.

Now we describe time-dependent variation, i.e. perturbations, of all variables according to

$$r(m,t) = r_0(m)\left[1 + x(m)e^{i\omega t}\right] ,$$
$$P(m,t) = P_0(m)\left[1 + p(m)e^{i\omega t}\right] ,$$
$$T(m,t) = T_0(m)\left[1 + \vartheta(m)e^{i\omega t}\right] ,$$
$$l(m,t) = l_0(m)\left[1 + \lambda(m)e^{i\omega t}\right] ,$$

where the absolute values of x, p, ϑ and λ are $\ll 1$.

These time-dependent variables obey time-dependent stellar structure equations.

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For example, insert

$$r(m,t) = r_0(m)\left[1 + x(m)e^{i\omega t}\right] ,$$
$$P(m,t) = P_0(m)\left[1 + p(m)e^{i\omega t}\right] ,$$

into

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \quad -\partial^2 r/\partial t^2 (4\pi r^2)^{-1} \quad ,$$

and linearize all (small in amplitude) perturbations, i.e. *p* and *x*, leads to

$$P_0'(1 + pe^{i\omega t}) + P_0 p' e^{i\omega t} = -\frac{Gm}{4\pi r_0^4}(1 - 4xe^{i\omega t}) + \frac{\omega^2}{4\pi r_0} x e^{i\omega t}$$

$P_0' := \mathrm{d}P_0/\mathrm{d}m$ $= -Gm/(4\pi r_0^4)$

$p' := \mathrm{d}p/\mathrm{d}m$

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relative density perturbation

With $\varrho \sim P^\alpha T^{-\delta}$ $\longrightarrow$ $d = \alpha p - \delta\vartheta$ the linearized perturbation equations are

$$p' = -\frac{P_0'}{P_0}\left[p + \left(4 + \frac{r_0^3}{Gm}\omega^2\right)x\right] ,$$
$$x' = -\frac{1}{4\pi r_0^3 \varrho_0}(3x + \alpha p - \delta\vartheta) .$$
$$\lambda' = -\frac{\varepsilon_0}{l_0}(\lambda - \varepsilon_P p - \varepsilon_T \vartheta) - i\omega\frac{P_0 \delta}{l_0 \varrho_0}\left(\frac{\vartheta}{\nabla_{ad}} - p\right) ,$$
$$\vartheta' = \frac{P_0'}{P_0}\nabla_{rad}[\kappa_P p + (\kappa_T - 4)\vartheta + \lambda - 4x] .$$

Time-dependent stellar-structure equations

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \varrho} ,$$
$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \partial^2 r/\partial t^2 (4\pi r^2)^{-1}$$
$$\frac{\partial l}{\partial m} = \varepsilon_n - \varepsilon_\nu - c_P\frac{\partial T}{\partial t} + \frac{\delta}{\varrho}\frac{\partial P}{\partial t} ,$$
$$\frac{\partial T}{\partial m} = -\frac{GmT}{4\pi r^4 P}\nabla ,$$

Stellar Stability
Dynamical Stability

For $\tau_{hydr} \ll \tau_{adj}$, the temperature of the matter changes almost adiabatically. (constant entropy s)

$$p' = -\frac{P_0'}{P_0}\left[p + \left(4 + \frac{r_0^3}{Gm}\omega^2\right)x\right] ,$$
$$x' = -\frac{1}{4\pi r_0^3 \varrho_0}\left(3x + \frac{1}{\gamma_1}p\right) .$$

$$d = \frac{1}{\gamma_1}p$$

Eigenvalue problem for eigenvalues ω_n^2 which are real.

- stability demands that for *all* eigenvalues $\omega_n^2 > 0$
- even a *single* eigenvalue with $\omega_n^2 < 0$ is sufficient for instability.
- The sign of ω_n^2 depends on the value of $\gamma_1 = \dfrac{1}{\alpha - \delta\nabla_{ad}}$.

Stellar Stability
Dynamical Stability

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Consider homologous changes of sphere $r = r(m)$ in hydrostatic equilibrium:

$$P = \int_m^M \frac{Gm}{4\pi r^4} dm \, .$$

Compression and homology varies **right-hand-side** as

value after compression ↖ $(R'/R)^{-4}$

remember: $\frac{\dot P}{P} = -4\frac{\dot r}{r}$

Adiabaticity and homology demands **left-hand-side** to vary as

$$(\varrho'/\varrho)^{\gamma_{ad}} = (R'/R)^{-3\gamma_{ad}}$$

remember: $\frac{\dot\varrho}{\varrho} = -3\frac{\dot r}{r}$

Stellar Stability
Dynamical Stability

$$P = \int_m^M \frac{Gm}{4\pi r^4} dm \, .$$

$(R'/R)^{-3\gamma_{ad}}$ ↙ ↘ $(R'/R)^{-4}$

if $\gamma_{ad} > 4/3$ the pressure on the left-hand side increases stronger with the contraction than the weight on the right: The resulting force is directed outwards and the star will move back towards equilibrium: it is dynamically stable.

For $\gamma_{ad} < 4/3$ the weight increases stronger than the pressure and the star would collapse after the initial compression (dynamical instability).

$\gamma_{ad} = 4/3$ the compression leads again to hydrostatic equilibrium: neutral equilibrium.

The critical value 4/3 depends strongly on spherical symmetry and Newtonian gravitation, i.e. radiation pressure, ionization, general relativity and rotation.

Stellar Stability
Dynamical Stability

Dynamical stability:

adiabatic change: $P \propto \rho^{\gamma_{ad}}$

homologous compression: $\rho \propto R^{-3}$

$\Big\} \; P \propto R^{-3\gamma_{ad}}$

From hydrostatic equilibrium: $P = \int_m^M \frac{Gm}{4\pi r^4} dm$

adiabatic **and** homologous change ↙ $R^{-3\gamma_{ad}}$ $\propto$ R^{-4} ↖ homologous change

➡ for $\gamma_{ad} \geq 4/3$ pressure on left-hand-side increases faster than "weight" on the right ⟶ stable.

for $\gamma_{ad} < 4/3$ ⟶ unstable.

Stellar Stability
The Gravothermal Specific Heat

Consider small sphere around stellar centre with $P \sim P_c$ and $\rho \sim \rho_c$

Then any mass shell of radius r after expansion has the radius $r + dr = r(1+x)$

Homologous change: $\dfrac{d\varrho_c}{\varrho_c} = -3x, \quad p_c := \dfrac{dP_c}{P_c} = -4x \, .$

EOS change: $\dfrac{d\varrho_c}{\varrho_c} = \alpha p_c - \delta\vartheta_c \, , \qquad (\vartheta_c := dT_c/T_c)$

Elimination of $d\varrho_c/\varrho_c$ $\quad p_c = \dfrac{4\delta}{4\alpha - 3}\vartheta_c \, .$

first law of thermodynamics: the heat dq per mass unit added to the central sphere is

$$dq = du + Pdv = c_P T_c(\vartheta_c - \nabla_{ad} p_c) := c^* T_c \vartheta_c \, ,$$

Stellar Stability

The Gravothermal Specific Heat

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Gravothermal specific heat:
$$c^* = c_P \left(1 - \nabla_{ad} \frac{4\delta}{4\alpha - 3}\right) \ . \tag{25.29}$$

For an ideal monatomic gas ($a = \delta = 1, \nabla_{ad} = 2/5$), as we have approximately in the central region of the Sun, one finds from (25.29) that $c^* < 0$.

This is fortunate, since if in the Sun the nuclear energy generation is accidentally enhanced for a moment ($dq > 0$), then $dT < 0$, the region cools, thereby reducing the overproduction of energy immediately. Therefore the negative specific heat acts as a stabilizer.

Note: energy conservation is fulfilled because of (mechanical expansion) *PdV* term, i.e. the whole star expands while the centre cools.

For a nonrelativistic degenerate gas ($\delta \to 0, \alpha \to 3/5$) equation (25.29) gives $c^* > 0$: The addition of energy to the central sphere heats up the matter, which can lead to thermal runaway. ⟶ **shell (burning) instability even for ideal gas!**