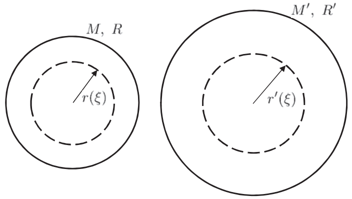
Homology relations

- Similar stellar models computed with the same assumptions (parameters and material functions) may differ in such a way that these differences could be described by simple analytical expressions.
- Only 'one' stellar model would then be necessary to compute and from which new stellar models could be derived using simple analytical expressions.
- Models with such similar properties are called '*homologous stars*'.

Homologous stars:



relative mass value (homologous masses):

$$\xi := \frac{m}{M} = \frac{m'}{M'}$$

homology condition (points, for any ξ):

$$\frac{r(\xi)}{r'(\xi)} = \frac{R}{R'} = \frac{r(0.7)}{r'(0.7)} = \ldots$$

… this ratio for homologous mass shells is constant throughout the stars.

Homology relations

- Since all models have to fulfil the stellar-structure equations, the transition from one to the next model has consequences for all variables.
- Consider two homologous stars with masses *M* & *M'* and of two different, but **homogeneous** (i.e. constant with radius) **compositions** represented by μ & μ'.

Basic parameters: $x = M/M'$; $y = \mu/\mu'$.

We use following 'ansatz' for homologous mass values

$$\frac{r}{r'} = z = \frac{R}{R'}\ ;\quad \frac{P}{P'} = p = \frac{P_c}{P'_c}\ ;\quad \frac{T}{T'} = t = \frac{T_c}{T'_c}\ ;\quad \frac{l}{l'} = s = \frac{L}{L'}\ ,$$

where *z*, *p*, *t*, and *s* **have the same values for any** $\xi := m/M = m'/M'$, and the subscript c denotes central values.

Homology relations

- For homologous main-sequence stars we can neglect the inertia and time-dependent terms in the equations for stellar structure, i.e.

using $\xi := m/M = m'/M'\ ;\ x = M/M';\quad y = \mu/\mu'\ ;\quad d = \rho/\rho'\ ;\ k = \kappa/\kappa'\ ;\ e = \varepsilon/\varepsilon'\ ;$

and $\frac{r}{r'} = z = \frac{R}{R'}\ ;\quad \frac{P}{P'} = p = \frac{P_c}{P'_c}\ ;\quad \frac{T}{T'} = t = \frac{T_c}{T'_c}\ ;\quad \frac{l}{l'} = s = \frac{L}{L'}\ ,$

independent of ξ

$$\frac{dr}{d\xi} = c_1 \frac{M}{r^2 \varrho}\ ,\qquad c_1 = \frac{1}{4\pi}\ ,$$
$$\frac{dP}{d\xi} = c_2 \frac{\xi M^2}{r^4}\ ,\qquad c_2 = -\frac{G}{4\pi}\ ,$$
$$\frac{dl}{d\xi} = \varepsilon M\ ,$$
$$\frac{dT}{d\xi} = c_4 \frac{\kappa l M}{r^4 T^3}\ ,\qquad c_4 = -\frac{3}{64\pi^2 ac}\ .$$

stellar model (r, P, T, l) (for *M*, μ)

$r, P, T, l \Longrightarrow r', P', T', l'$

$$\frac{dr'}{d\xi} = c_1 \frac{M'}{r'^2 \varrho'} \left[\frac{x}{z^3 d}\right] ,\quad = 1$$
$$\frac{dP'}{d\xi} = c_2 \frac{\xi M'^2}{r'^4} \left[\frac{x^2}{z^4 p}\right] ,\quad = 1$$
$$\frac{dl'}{d\xi} = \varepsilon' M \left[\frac{ex}{s}\right] ,\quad = 1$$
$$\frac{dT'}{d\xi} = c_4 \frac{\kappa' l' M'}{r'^4 T'^3} \left[\frac{ksx}{z^4 t^4}\right] .\quad = 1$$

stellar model (r', P', T', l') (for *M'*, μ')

Homology relations

The stellar equations must be obeyed by both sets of dependent variables, i.e, (*r, P, T, l*) AND (*r', P', T', l'*), and consequently the factors

$$\frac{x}{z^3 d} = 1\ ,\quad \frac{x^2}{z^4 p} = 1,\quad \frac{ex}{s} = 1,\quad \frac{ksx}{z^4 t^4} = 1\ .$$

$$\frac{\varrho}{\varrho'} = \frac{M/M'}{(R/R')^3},\quad \frac{P}{P'} = \frac{(M/M')^2}{(R/R')^4}\ .$$

i.e. for all *homologous points* ξ, the density changes simply as the mean density for the whole star, while *P* varies like $M^2 R^{-4}$.

Homology relations

- For the remaining solutions we still need to **represent the material functions by power laws**:

$$\varrho \sim P^{\alpha} T^{-\delta} \mu^{\varphi}, \quad \varepsilon \sim \varrho^{\lambda} T^{\nu}, \quad \kappa \sim P^{a} T^{b} \ ,$$

with which we can derive from the definitions for *z, p, t, s, d, e*, and *k*
$d = \rho/\rho'$; $k = \kappa/\kappa'$; $e = \varepsilon/\varepsilon'$;

$$d = p^{\alpha} t^{-\delta} y^{\varphi}, \quad e = p^{\lambda\alpha} t^{\nu-\lambda\delta} y^{\lambda\varphi}, \quad k = p^{a} t^{b} \ .$$

We further need to represent *z, p, t, s* **in terms of *x=M/M'*** and ***y=μ/μ'***, in order to describe their dependence on the basic parameters *M* and μ:

$$z = x^{z_1} y^{z_2} \ ; \quad p = x^{p_1} y^{p_2} \ ; \quad t = x^{t_1} y^{t_2} \ ; \quad s = x^{s_1} y^{s_2} \ .$$

We need 8 linear equations for solving the 8 exponents.

Homology relations

require 8 linear equations for solving the 8 exponents obtained by inserting

$$z = x^{z_1} y^{z_2} \ ; \quad p = x^{p_1} y^{p_2} \ ; \quad t = x^{t_1} y^{t_2} \ ; \quad s = x^{s_1} y^{s_2}$$
$$d = p^{\alpha} t^{-\delta} y^{\varphi}, \quad e = p^{\lambda\alpha} t^{\nu-\lambda\delta} y^{\lambda\varphi}, \quad k = p^{a} t^{b}$$

$$\frac{x}{z^3 d} = 1 \ , \quad \frac{x^2}{z^4 p} = 1, \quad \frac{ex}{s} = 1, \quad \frac{ksx}{z^4 t^4} = 1$$

4 conditions with $x^u \cdot y^v = 1$
$u = \sum (z,p,t,s)_1 = 0$
$v = \sum (z,p,t,s)_2 = 0$
because rhs is independent of *x,y*

Homology relations

require 8 linear equations for solving the 8 exponents obtained by inserting

$$z = x^{z_1} y^{z_2} \ ; \quad p = x^{p_1} y^{p_2} \ ; \quad t = x^{t_1} y^{t_2} \ ; \quad s = x^{s_1} y^{s_2}$$
$$d = p^{\alpha} t^{-\delta} y^{\varphi}, \quad e = p^{\lambda\alpha} t^{\nu-\lambda\delta} y^{\lambda\varphi}, \quad k = p^{a} t^{b}$$

$$\frac{x}{z^3 d} = 1 \ , \quad \frac{x^2}{z^4 p} = 1, \quad \frac{ex}{s} = 1, \quad \frac{ksx}{z^4 t^4} = 1$$

4 conditions with $x^u \cdot y^v = 1$
$u = \sum (z,p,t,s)_1 = 0$
$v = \sum (z,p,t,s)_2 = 0$
because rhs is independent of *x,y*

(1)–(4):

$$\begin{pmatrix} -3 & -\alpha & \delta & 0 \\ -4 & -1 & 0 & 0 \\ 0 & \lambda\alpha & (\nu-\lambda\delta) & -1 \\ -4 & a & (b-4) & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ p_1 \\ t_1 \\ s_1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \\ -1 \end{pmatrix}$$

(1), (2): mechanical equ.
(3): energy equ. (ε)
(4): energy transfer

(5)–(8):

$$\begin{pmatrix} -3 & -\alpha & \delta & 0 \\ -4 & -1 & 0 & 0 \\ 0 & \lambda\alpha & (\nu-\lambda\delta) & -1 \\ -4 & a & (b-4) & 1 \end{pmatrix} \begin{pmatrix} z_2 \\ p_2 \\ t_2 \\ s_2 \end{pmatrix} = \begin{pmatrix} \varphi \\ 0 \\ -\lambda\varphi \\ 0 \end{pmatrix} \ .$$

Homology relations

The solution of the eight linear equations is:

$$z_1 = \frac{1}{2}(1+A), \quad p_1 = -2A \ ,$$
$$t_1 = \frac{1}{2\delta}[1 + (3-4\alpha)A] \ ,$$
$$s_1 = 1 + \frac{4-b}{2\delta} + \left[2 + 2a + \frac{3-4\alpha}{2\delta}(4-b)\right] A \ ,$$

$$z_2 = \varphi B, \quad p_2 = -4\varphi B, \quad t_2 = \frac{\varphi}{\delta}[1 + (3-4\alpha)B] \ ,$$
$$s_2 = \frac{\varphi}{\delta}(4-b) + \varphi\left[4 + 4a + \frac{3-4\alpha}{\delta}(4-b)\right] B \ ,$$

$$A = \left[\frac{4\delta(1+a+\lambda\alpha)}{\nu+b-4-\lambda\delta} + 4\alpha - 3\right]^{-1}, \quad B = A\left(1 - \frac{\lambda\delta}{\nu+b-4}\right)^{-1} \ .$$

Homology relations

Application to simple material functions: δ=0 → $\rho \propto P^\alpha \mu^\varphi$ → polytropic EOS

→ polytropic index $n = \alpha/(1-\alpha)$

- first (decoupled) 2 equations (of the eight), which represent the mechanical part, can be solved independently: we have

$$A = \frac{1}{4\alpha - 3} \quad ; \quad z_1 = \frac{2\alpha - 1}{4\alpha - 3} .$$

And from $z = x^{z_1} y^{z_2}$ i.e. $R/R' = (M/M')^{z_1}$ assuming $y = 1$, i.e. for *homologous stars* with identical composition μ

$$R \propto M^{z_1}$$

e.g., for a non-relativistic, degenerate electron gas with $\alpha = 3/5$ we get

$$z_1 = -1/3. \longrightarrow R \propto M^{-1/3}$$

as already discussed in the polytrope section with fixed *K* (and *n*=3/2).

Homology relations

Application to simple material functions: $\overbrace{\alpha = 1;\ \delta = 1;\ \varphi = 1;}^{\text{ideal gas}}\ \overbrace{a = 0;\ b = 0}^{\text{constant opacity}}$.

from solutions of 8 linear equations →

$$z_1 = \frac{\nu + \lambda - 2}{\nu + 3\lambda}, \qquad z_2 = \frac{\nu - 4}{\nu + 3\lambda},$$
$$p_1 = 2 - 4z_1, \qquad p_2 = -4z_2,$$
$$t_1 = 1 - z_1, \qquad t_2 = 1 - z_2,$$
$$l/l' =: s := x^{s_1} y^{s_2} \rightarrow s_1 = 3, \qquad s_2 = 4 .$$

→ independent of ν and λ $(\varepsilon = \rho^\lambda T^\nu)$

$$\frac{L}{L'} = \left(\frac{M}{M'}\right)^3 \left(\frac{\mu}{\mu'}\right)^4 .$$

The model has to adjust so that the energy sources (ε) provide the required luminosity, the latter only determined by the hydrostat. equ., EOS and radiative *transport* (*opacity!*), but not the energy equation.

All other exponents depend on ν and λ

e.g. z_1 & z_2 describe: $$\frac{R}{R'} = \left(\frac{M}{M'}\right)^{z_1} \left(\frac{\mu}{\mu'}\right)^{z_2} .$$

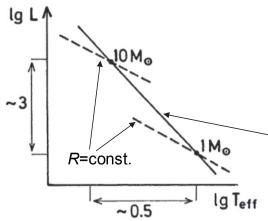
Homology relations

All other exponents depend on ν and λ

e.g. z_1 & z_2 describe: $$\frac{R}{R'} = \left(\frac{M}{M'}\right)^{z_1} \left(\frac{\mu}{\mu'}\right)^{z_2} .$$

z_1 is positive for all relevant combinations of ν and λ but < 1, i.e. *R* increases only slightly with *M* (values from 8 equations and for λ=1, $\alpha=\delta=\varphi$=1, *a* = *b* = 0):

	pp		CNO	
ν:	4	5	15	18
z_1	0.43	0.5	0.78	0.81
z_2	0	0.13	0.61	0.67



From previous 2 equ. we have $R \propto L^{z_1/3}$ (μ=const.) insert into $\sigma T_{\rm eff}^4 = L/4\pi R^2$

→ locus in Hertzsprung-Russell diagram for homologous M-S star

$$\lg L = \frac{12}{3 - 2z_1} \lg T_{\rm eff} + \text{constant} .$$

For an average value $z_1 = 0.6$, the slope is 6.67.

Homology relations

$$z_1 = \frac{\nu + \lambda - 2}{\nu + 3\lambda}, \qquad z_2 = \frac{\nu - 4}{\nu + 3\lambda},$$
$$p_1 = 2 - 4z_1, \qquad p_2 = -4z_2,$$
$$t_1 = 1 - z_1, \qquad t_2 = 1 - z_2,$$
$$s_1 = 3, \qquad s_2 = 4 .$$

$$t = \frac{T}{T'} \propto x^{t_1} = \left(\frac{M}{M'}\right)^{1 - z_1} \qquad \left(\frac{M}{M'}\right)^{z_1} = \frac{R}{R'}$$

$$\longrightarrow T \propto \frac{M}{R}$$

$E_{\rm kin} \propto kT$; $E_{\rm g} = -G \int (m/r) {\rm d}m \propto -M^2/R$

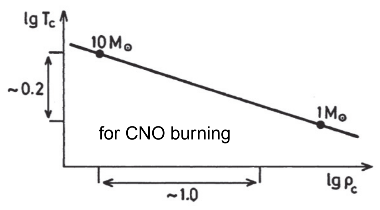
...reflects $E_{\rm kin} = E_{\rm pot}$ (virial theorem)

Homology relations

central values T_c and ρ_c

$$t = \frac{T_c}{T_c'} \propto \left(\frac{M}{M'}\right)^{1-z_1} \qquad \rightarrow T_c \propto M^{1-z_1}$$

$$\frac{\rho_c}{\rho_c'} \propto \frac{M/M'}{(R/R')^3} \propto \frac{M/M'}{(M/M')^{3z_1}} \qquad \rightarrow \rho_c \propto M^{1-3z_1}$$

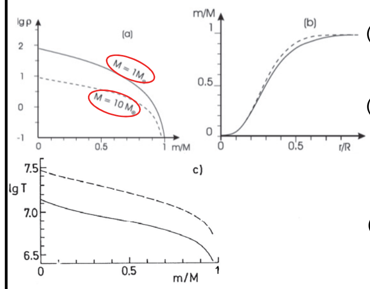


for CNO burning:

$T_c \propto M^{0.2}$ ... increases slowly with *M*

$\rho_c \propto M^{-1.4}$... decreases with *M* !

The (zero-age) Main Sequence: interior solutions



(a) $1M_\odot$: ρ increases by ~ 10^9 from surface to centre. For $10M_\odot$ ρ_c is smaller by a factor of ~10.

(b) Mass elements are increasingly concentrated towards the centre. $1M_\odot$: inner ~30% of *R* (3% in *V*) contain 50% of *M*; outer ~50% of *R* (88% in *V*) contain only 15% of *M*.

(c) $1M_\odot$: temperature *T* increases by a factor of ~ 2400 from photosphere to centre. $10M_\odot$: *T* is ~2 times higher than for $1M_\odot$.

Homology relations

The Role of the Equation of State

The procedure by which the homology solutions were obtained shows that their existence rests entirely on the fact that the right-hand sides of (20.5) contain only products of the variables, but no sums.

But (e.g. EOS destroys this, i.e.): $\quad P = \Re\varrho T/\mu + aT^4/3.$

$$\varrho \sim (\mu\beta)\frac{P}{T}, \quad \beta = \frac{P_{\rm gas}}{P} = \frac{1-P_{\rm rad}}{P}.$$

constant (w.r.t. to *r*)

→ same effect as that of μ and we would find $R \sim \beta^{z_2}, P \sim \beta^{p_2}, T \sim \beta^{t_2}, L \sim \beta^{s_2}$.

$$1-\beta = \frac{P_{\rm rad}}{P} \sim \frac{T^4}{P} \sim \frac{M^{4t_1}}{M^{p_1}}\frac{\beta^{4t_2}}{\beta^{p_2}}, \qquad \frac{1-\beta}{\beta^4} \sim M^2.$$

monatomic gas
$s_1 = 3$,
$s_2 = 4$

$L \sim M^{s_1}\beta^{s_2} \quad \rightarrow \quad$ For $\beta \ll 1$ (large $P_{\rm rad}$) $\beta \sim M^{-1/2} \rightarrow L \sim M^{s_1 - s_2/2} = M.$

M–L relation becomes less steep.

Homology relations

Homologous contraction

- consider contracting polytrope with constant index *n* .
- mass *M* stays constant but radius *R*(*t*) is changing with time *t* .
- each **mass element *m(z)*** remains at homologous point *z*(ξ) = *r*(*t*)/*R*(*t*), which is **independent of *t* !**
- homologous mass shells (ξ = *m*/*M* = constant during contraction) have therefore same value *m*, because normalizing factor *M* remains constant, but any such shell's radius *r(t)* changes by the rate $\dot r = \partial r/\partial t$, i.e. with $r' = r + \dot r \Delta t$ and

$$\frac{r'}{r} = 1 + \frac{\dot r}{r}\Delta t = \frac{R'}{R} \text{ = constant (in space) for homologous contraction}$$

$$\longrightarrow \boxed{\frac{\dot r}{r} = \frac{\dot R}{R} \text{ = constant}},$$

or $$\frac{\partial}{\partial m}\left(\frac{\partial \ln r}{\partial t}\right) = 0\,.$$

Homology relations

Homologous contraction

for homologous contraction $\frac{\dot{r}}{r} = \frac{\dot{R}}{R}$ = constant

or $\frac{\partial}{\partial m}\left(\frac{\partial \ln r}{\partial t}\right) = 0 \,.$

From mass conservation equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial r}{\partial m}\right) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi r^3 \varrho}\right) = \frac{1}{4\pi r^3 \varrho}\left(-3\frac{\dot{r}}{r} - \frac{\dot{\varrho}}{\varrho}\right) = 0 \quad \longrightarrow \quad \frac{\dot{\varrho}}{\varrho} = -3\frac{\dot{r}}{r} \,.$$

From hydrostatic equilibrium: $P = \int_m^M \frac{Gm}{4\pi r^4} dm$

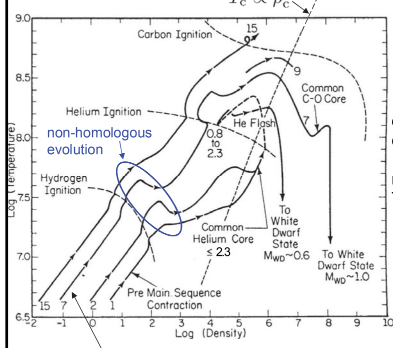
$$\dot{P} = \int_m^M \frac{\partial}{\partial t}\left(\frac{1}{r^4}\right)\frac{Gm}{4\pi} dm = -4\frac{\dot{r}}{r}\int_m^M \frac{Gm}{4\pi r^4} dm \quad \longrightarrow \quad \frac{\dot{P}}{P} = -4\frac{\dot{r}}{r} \,.$$

From EOS: $\dot{\varrho}/\varrho = \alpha \dot{P}/P - \delta \dot{T}/T \quad \longrightarrow \quad \frac{\dot{T}}{T} = -\frac{4\alpha - 3}{\delta}\frac{\dot{r}}{r} \,.$

Later phases of core evolution

Evolution of central regions

$T_c \propto \rho_c^{2/3}$



$T_c \propto \rho_c^{1/3}$

$\mathrm{d}\ln T_c = (4\alpha - 3)/(3\delta)\,\mathrm{d}\ln\rho$

in non-degenerate regions $T \sim \rho^{1/3}$;

non-homologous evolution because of structure changes due to radiation – convective core.

each contraction with $T \sim \rho^{1/3}$ brings centre closer to e^- degeneracy; in degenerate regions T_c no longer increases, and the next burning is not reached by further contraction (if at all). This is more likely the case for low-mass stars.

This happens the earlier in nuclear history the closer to degeneracy star has been @ ZAMS, i.e. which nuclear cycle is completed before star develops degenerate core depends on the stellar mass *M*.

Homology relations

Homologous contraction

for homologous contraction $\frac{\dot{r}}{r} = \frac{\dot{R}}{R}$ = constant

or $\frac{\partial}{\partial m}\left(\frac{\partial \ln r}{\partial t}\right) = 0 \,.$

From mass conservation equation:

$$\frac{\partial}{\partial t}\left(\frac{1}{r}\frac{\partial r}{\partial m}\right) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi r^3 \varrho}\right) = \frac{1}{4\pi r^3 \varrho}\left(-3\frac{\dot{r}}{r} - \frac{\dot{\varrho}}{\varrho}\right) = 0 \quad \longrightarrow \quad \frac{\dot{\varrho}}{\varrho} = -3\frac{\dot{r}}{r} \,.$$

From hydrostatic equilibrium: $P = \int_m^M \frac{Gm}{4\pi r^4} dm$

$$\dot{P} = \int_m^M \frac{\partial}{\partial t}\left(\frac{1}{r^4}\right)\frac{Gm}{4\pi} dm = -4\frac{\dot{r}}{r}\int_m^M \frac{Gm}{4\pi r^4} dm \quad \longrightarrow \quad \frac{\dot{P}}{P} = -4\frac{\dot{r}}{r} \,.$$

$$\varepsilon_g = c_P T\left(\nabla_{ad}\frac{\dot{P}}{P} - \frac{\dot{T}}{T}\right) = c_P T\left(-4\nabla_{ad} + \frac{4\alpha - 3}{\delta}\right)\frac{\dot{R}}{R} \overset{\substack{\text{monatomic gas} \\ (\nabla_{ad} = 2/5,\, \alpha = \delta = 1)}}{=} -\frac{3}{5}c_P T\frac{\dot{R}}{R} \,.$$