## Nuclear energy production

Aim: evaluate energy-generation rate per unit mass $\epsilon$ .

Sun: $\langle\epsilon\rangle \simeq 10\ \mathrm{erg\ s^{-1}\ g^{-1}}$ (check $L_\odot/M_\odot$ , $\epsilon_{\text{human}} \simeq 10^4\ \mathrm{erg\ s^{-1}\ g^{-1}}$ )

energy-generation rate produced from fusion of two nuclei a + A:

$$\underbrace{\epsilon_{\mathrm{aA}} = Q_{\mathrm{aA}}}_{\text{energy released per reaction}} \times \underbrace{r_{\mathrm{aA}}}_{\text{reaction rate per unit volume}} \times \frac{1}{\rho}$$

energy released per reaction

reaction rate per unit volume (includes cross section $\sigma(E)$ and velocity distribution $f(v)\mathrm{d}v$ )

## Nuclear energy production

- Nuclear reactions in sun take place within inner ~10% of total solar mass.
- Estimation of energy generation by "hydrogen burning" reactions

$$4\,{}^1\mathrm{H} \to {}^4\mathrm{He} + 2\mathrm{e}^+ + 2\nu_\mathrm{e}\ .$$

- mass loss:

$$\begin{array}{ll} \mathrm{H}: 4\times 1.008 = & 4.032\ \mathrm{m_u} \\ \mathrm{He}: & -4.003\ \mathrm{m_u} \\ 2\mathrm{e}^-: & -0.001\ \mathrm{m_u} \\ \hline \Delta m: & 0.028\ \mathrm{m_u} \end{array} \quad \to 0.028/4 = 0.7\%$$

$$\sim 26.21\ \mathrm{MeV}$$

$$(1\ \mathrm{eV} = 1.6020\times 10^{-12}\ \mathrm{erg})$$

- nuclear time scale: $t_{\mathrm{nuc}} := \frac{\Delta m\tilde{c}^2}{\mathrm{L}_\odot} = 7\times 10^{-3}\frac{\mathrm{M}_\odot}{10}\frac{\tilde{c}^2}{\mathrm{L}_\odot}$

$$t_{\mathrm{nuc}} = 10^{10}\mathrm{years}\left(\frac{M}{\mathrm{M}_\odot}\right)\left(\frac{L}{\mathrm{L}_\odot}\right)^{-1}$$

## Nuclear energy production

**Energy $Q_{\mathrm{aA}}$ by each reaction**

- Most stars live from so-called thermonuclear fusion, where due to thermal motion lighter nuclei fuse to form heavier elements.
- During fusion process, some of the mass (e.g. 0.7% for hydrogen burning) of original nuclei has been converted into energy according to $E = \Delta m\tilde{c}^2$.
- Mass loss $\Delta m$ origins in different binding energies $E_\mathrm{B}$ of the involved nuclei.
- $E_\mathrm{B}$ is energy required to separate nucleons (protons & neutrons) against their mutual attraction of the strong, short-range forces, or the gain if nucleons are brought together (within 10$^{-12}$ cm) from infinity.

$$E_\mathrm{B} = [\underbrace{Nm_\mathrm{u}}_{\text{total rest mass of neutrons}} + \underbrace{Zm_\mathrm{u}}_{\text{total rest mass of protons}} - \underbrace{M_{\mathrm{nuc}}}_{\text{mass of nucleus}}]\,\tilde{c}^2$$

Integer atomic weight: $A = N + Z$

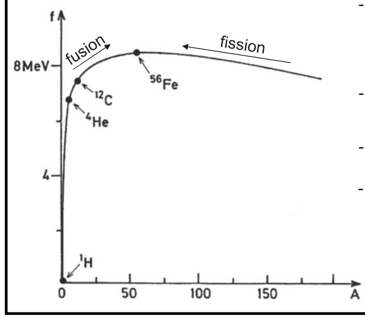
## Nuclear energy production

**Energy $Q_{\mathrm{aA}}$ by each reaction**

- for comparing nuclei, better $E_\mathrm{B}$ per nucleon: $f = \frac{E_\mathrm{B}}{A}$

$E_\mathrm{B} = [Nm_\mathrm{u} + Zm_\mathrm{u} - M_{\mathrm{nuc}}]\,\tilde{c}^2$

Integer atomic weight: $A = N + Z$




- for *A*<56, *f*↑↑, because of short-range, strong-forces affecting only nucleons in its immediate neighbourhood only (geometrical effect: surface increases with *r* slower than volume, i.e. with *A*).
- for *A*>56, *f*↓↓ , because of repulsive Coulomb forces between protons, which are far-reaching.
- $^{56}$Fe most tightly bound nuclei (smallest *m*/nuclei).
- any reaction bringing resulting nuclei close to $f_{\max}$ will be exothermic:

  (a) for A<56 by fusion (e.g. stellar cores)
  (b) for A<56 by fission (e.g. radioactivity)

## Nuclear energy production

**Energy $Q_{\mathrm{aA}}$ by each reaction**

- Consider general reaction:

$$A + a \rightarrow Y + y$$

Short notation:

$$A(a,y)Y$$

- The energy $Q_{\mathrm{aA}}$ released in this reaction is:

$$Q_{\mathrm{aA}} = \bar{c}^2[m(A) + m(a) - m(Y) - m(y)]$$

- Since loss of mass is generally very small, more convenient to use *mass excess*

$$\Delta m = m - m_{\mathrm{u}}(\underset{\text{protons}}{\mathcal{Z}} + \underset{\text{neutrons}}{\mathcal{N}}) ,$$

$$Q_{\mathrm{aA}} = \bar{c}^2[\Delta m(A) + \Delta m(a) - \Delta m(Y) - \Delta m(y)] .$$

atomic mass excesses $\Delta m$ available in form of tables (e.g. Clayton 1968).

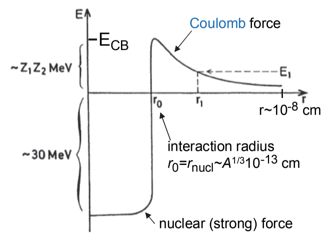
## Nuclear energy production

**The cross sections $\sigma(E)$** $(\mathrm{barn} \;=\; 10^{-24}\,\mathrm{cm}^2)$

- $\sigma(E)$ defined as an area, which describes the relation between *probability p* of a reaction and a geometrical area.

- reaction between nuclei is caused by the *strong forces*, acting between protons and neutrons (range limited to extent of nucleus).

- for a reaction to occur, nuclei must be brought so close together that they touch each other → requires to overcome ***Coulomb repulsion*** between them.




- energy to overcome Coulomb repulsion,i.e. height of Coulomb barrier

$$E_{\mathrm{CB}} \simeq \frac{Z_1 Z_2 e^2}{r_0} \simeq Z_1 Z_2 \,\boxed{\mathrm{MeV}}$$

- however, average kinetic energy of particles

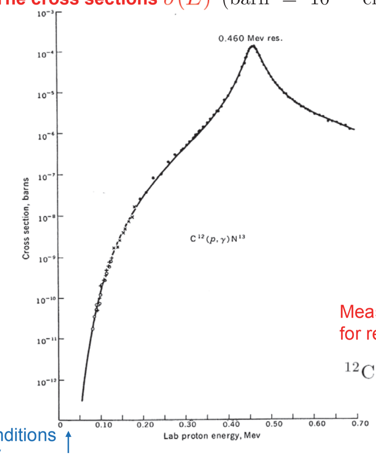
$$\langle E_{\mathrm{kin}} \rangle = \frac{3}{2} kT \simeq 130\,\mathrm{eV}\; T/10^6 \simeq \boxed{\mathrm{keV}}$$

$$\langle E_{\mathrm{kin}} \rangle \simeq \frac{1}{1000} E_{\mathrm{CB}}$$

## Nuclear energy production

**The cross sections $\sigma(E)$** $(\mathrm{barn} \;=\; 10^{-24}\,\mathrm{cm}^2)$

$\sigma(E) \;\; (\mathrm{barn} \;=\; 10^{-24}\,\mathrm{cm}^2)$




Measured cross section for reaction

$^{12}\mathrm{C}(^1\mathrm{H},\gamma)\,^{13}\mathrm{N}$

Energy (MeV)

Stellar core conditions ~25 keV

## Nuclear energy production

**The cross sections $\sigma(E)$**

- reasonable extrapolation of $\sigma(E)$ to lower $E$ can be obtained by separating strongest $E$-dependent contributions caused by (quantum-mechanical) penetration probability through Coulomb barrier.

- quantum-mechanical effect found by G. Gamov (1928) is the '***tunnel-effect***', in which nuclei can penetrate (tunnel) the potential barrier with a *probability* $p_{\mathrm{G}}(E)$

$$p_{\mathrm{G}}(E) \propto \exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right) ; \qquad \propto \hbar v\,; \propto 1/Z_1 Z_2 \,.$$

relative velocity between the two nuclei: $v \sim E^{1/2}$

- an additional $E$- dependence enters in the 'geometrical extent' of the nuclei:

$$\pi\lambda^2 \propto p^{-2} \propto E^{-1} \qquad \lambda = \frac{\hbar}{p} \quad \text{... de Broglie wavelength}$$

geometrical cross section

## Nuclear energy production

### The cross sections $\sigma(E)$

- *cross-section* factor (astrophysical cross section) $S(E)$

$$\sigma(E) \propto \underbrace{S(E)}_{\text{"astrophysical" }\sigma} \times \underbrace{\frac{1}{E}}_{\text{geometrical factor}} \times \underbrace{p_G(E)}_{\text{Coulomb barrier}} \longrightarrow \sigma(E) \equiv S(E) \times \frac{1}{E} \times \exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right)$$

- here $E$ is energy of particles in the centre-of-mass system

$$E = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}v^2 = \frac{1}{2}\mathcal{A} m_u v^2 \qquad \mathcal{A} = \frac{\mathcal{A}_1\mathcal{A}_2}{\mathcal{A}_1+\mathcal{A}_2}$$

… reduced atomic weight

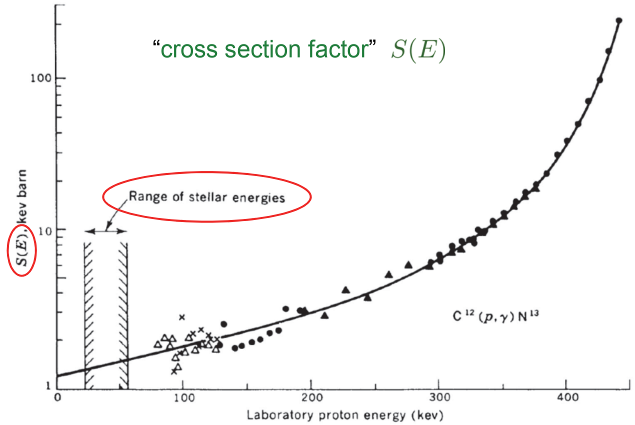
$$\longrightarrow \quad \sigma(E) = \frac{S(E)}{E} \times \exp\left(-bE^{-1/2}\right)$$

where $b = 31.291\, Z_1 Z_2 A^{1/2}\ \mathrm{keV}^{1/2}$ .

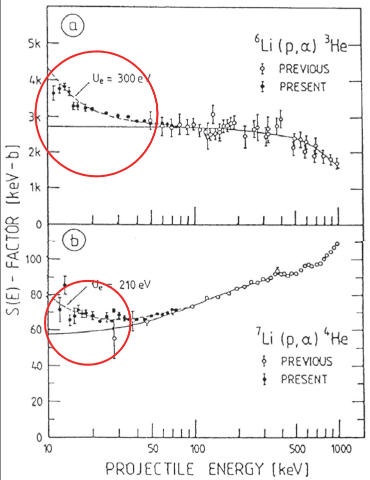
Reactions amongst nuclei of low charges are faster and are possible at low temperatures.

## Nuclear energy production

### The cross sections




## Nuclear energy production



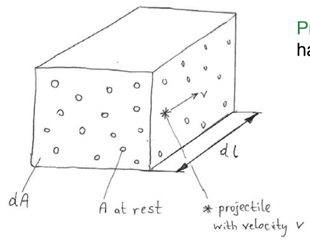

"astrophysical cross section" $S(E)$

-departure from (near) constant value at low E results from '**screening**' from electrons in the target $^6$Li or $^7$Li atoms, which reduces Coulomb potential around nucleus, and hence increases $S(E)$.

## Nuclear energy production

### The reaction rates $r_{aA}$

- we need relation between $\sigma(E)$ and total rate of reactions, between particles *a* and *A*




Probability that one particle *a*, the projectile, will have a collision along d*l* is

$$\frac{\text{sum of all } \sigma \text{ in the box}}{\mathrm{d}A},$$

i.e. $$\frac{\sigma n_A \mathrm{d}A\mathrm{d}l}{\mathrm{d}A} = \sigma n_A \mathrm{d}l\,.$$

- particle flux of projectile: $n_a\, v$

- reactions per unit time in whole box: $(\sigma n_A \mathrm{d}l)\overbrace{(n_a\, v\, \mathrm{d}A)}^{\text{projectiles passing } dA \text{ every second}} = \sigma\, v\, n_a n_A\, \mathrm{d}l\mathrm{d}A$

- reactions per unit time and volume:

$$r_{aA} = \sigma_{aA}(v)\, v\, n_a\, n_A \quad (\mathrm{cm^{-3}s^{-1}})$$

- special case *A*=*a* :

$$r_{AA} = \frac{1}{2}\sigma_{AA}(v)\, v\, n_A^2$$

## Nuclear energy production

### The reaction rates $r_{aA}$

- In reality $A$ is not at rest and there is a distribution of the relative velocities $v$. If $f(v)\mathrm{d}v$ is the fraction of pairs of particles with relative speed between $v$ & $v+\mathrm{d}v$

$$r_{aA} = \langle \sigma v \rangle n_a n_A \, ,$$

with

$$\langle \sigma v \rangle = \int_0^\infty v\sigma(E) f(v) \mathrm{d}v \, .$$

- We assume that both nuclei have M-B distribution, then (see Clayton 1968) the distribution of $v$ is also M-B

$$f(v)\mathrm{d}v = 4\pi \left(\frac{m}{2\pi k_\mathrm{B} T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_\mathrm{B}T}\right) v^2 \mathrm{d}v \quad ,$$

where

$$m = \mathcal{A} m_\mathrm{u} \qquad \text{with} \qquad \mathcal{A} = \frac{\mathcal{A}_1 \mathcal{A}_2}{\mathcal{A}_1 + \mathcal{A}_2}$$

$m$: reduced mass; $\mathcal{A}$: … reduced atomic weight.

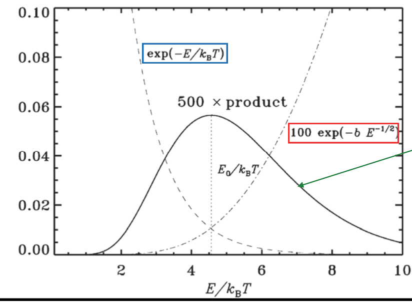
## Nuclear energy production

### The reaction rates $r_{aA}$

and we obtain

$$\langle \sigma v \rangle = \underbrace{\left(\frac{8}{m\pi}\right)^{1/2} \frac{1}{(k_\mathrm{B}T)^{3/2}}}_{\text{includes geometry of particles}} \int_0^\infty \underbrace{S(E)}_{\text{intrinsic nuclear reactions}} \exp\left(\underbrace{-\frac{E}{k_\mathrm{B}T}}_{\text{M-B factor of thermal energy}} \underbrace{- \frac{b}{E^{1/2}}}_{\text{Coulomb barrier (tunneling)}}\right) \mathrm{d}E \, .$$

$b = 31.291\, Z_1 Z_2 \mathcal{A}^{1/2}\ \mathrm{keV}^{1/2}$




competing effects

resulting in

Gamov peak maximum at

$$E_0 = \left(\frac{bk_\mathrm{B}T}{2}\right)^{2/3}$$

$^1\mathrm{H} + {}^1\mathrm{H}$ at $15 \times 10^6$ K

## Nuclear energy production

### The reaction rates $r_{aA}$

- analytical approximation of Gamov peak:

-we neglect variation of $S(E)$ over Gamov peak [$S(E) \to S(E_0)$], and approximate the function $\exp(-E/k_\mathrm{B}T - bE^{-1/2})$ by a Gaussian

$$\exp\left(-\frac{3E_0}{k_\mathrm{B}T}\right) \exp\left[-\left(\frac{E-E_0}{\Delta/2}\right)^2\right] \, ,$$

where the width $\Delta = 4(E_0 k_\mathrm{B} T/3)^{1/2}$ provides the same curvature at maximum. After some algebra we obtain

$$\langle \sigma v \rangle = \frac{8\sqrt{2}}{9\sqrt{3}} \frac{S(E_0)}{\sqrt{mb}} \eta^2 \exp(-\eta) \quad ,$$

with

$$\eta = \frac{3E_0}{k_\mathrm{B}T} = BT_6^{-1/3} \, , \qquad B = 42.487 (\mathcal{Z}_1^2 \mathcal{Z}_2^2 \mathcal{A})^{1/3} \, .$$

For fairly large $\eta$, $\langle \sigma v \rangle$ is a decreasing function of $\eta$: $\longrightarrow \langle \sigma v \rangle \downarrow\downarrow$ for $Z_1 Z_2 \uparrow\uparrow$ .

## Nuclear energy production

### The reaction rates $r_{aA}$

- in order to 'see' the $T$-dependence more clearly, we further simplify the expression over a limited $T$ range around $T=T_0$, defining $T_0$ such that $(3/2)kT_0 = E_0$ and adopt the power law

$$\langle \sigma v \rangle \simeq \langle \sigma v \rangle_0 \left(\frac{T}{T_0}\right)^n \, ,$$

where $\langle \sigma v \rangle_0$ is the value of $\langle \sigma v \rangle$ at $T = T_0$ and

$$n = \frac{\mathrm{d}\ln\langle \sigma v \rangle}{\mathrm{d}\ln T} = \frac{\mathrm{d}\ln\langle \sigma v \rangle}{\mathrm{d}\ln \eta} \frac{\mathrm{d}\ln \eta}{\mathrm{d}\ln T} = \frac{\eta - 2}{3} \, ,$$

evaluated at $T = T_0$.

Because $\eta \propto (Z_1 Z_2)^{3/2}$, the $T$-sensitivity of $\langle \sigma v \rangle$ increases strongly with $Z_1 Z_2$ :

e.g. $n$= 4……5 for H-burning in Sun
$n$=13…23 for CNO-cycle .

## Nuclear energy production
### The reaction rates $r_{aA}$

We may now write the reaction rates (reactions /s /cm³) as

$$r_{aA} = \langle\sigma v\rangle_0 n_a n_A \left(\frac{T}{T_0}\right)^n .$$

The energy generation rate per unit mass $\epsilon_{aA}$ is then

$$\epsilon_{aA} = Q_{aA} \times r_{aA} \times \frac{1}{\rho}$$

$$\boxed{\epsilon_{aA} = Q_{aA}\langle\sigma v\rangle_0 \frac{X_a X_A}{\mathcal{A}_A \mathcal{A}_a m_u^2}\rho\left(\frac{T}{T_0}\right)^n}$$

where we used

$$n_A = \frac{X_A \rho}{\mathcal{A}_A m_u}$$

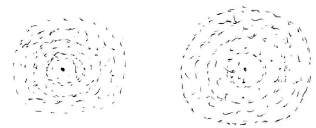
$X_A$, $X_a$ abundances by mass

Rate of change in $X_A$ due to reactions with nuclei of type $a$

$$\boxed{\frac{\mathrm{d}X_A}{\mathrm{d}t} = r_{aA}\frac{X_A}{n_A} = r_{aA}\frac{m_u \mathcal{A}_A}{\rho} = -\langle\sigma v\rangle_0 \frac{X_a X_A}{\mathcal{A}_a m_u}\rho\left(\frac{T}{T_0}\right)^n \equiv -\frac{X_A}{\tau_{aA}}}$$

## Nuclear energy production
### Electron screening

- repulsive Coulomb force is important for estimating $\epsilon_{aA}$.
- free e⁻ have influence on Coulomb force, i.e. $E_{CB}$.
- approaching particle will "feel" neutral conglomerate of target nucleus & surrounding e⁻ - cloud.
- e⁻ are attracted from nucleus of charge $+Ze$ → e⁻ have slightly ↑↑$n_e$ near nucleus, and ions are repelled, i.e. $n_i$ ↓↓ ⟶ clustering of e⁻ .



clustering of e⁻ around nucleus

## Nuclear energy production
### Electron screening

- particle density $n$ with charge '$q$' is modified in the presence of an electrostatic potential $\phi$ according to

$$n = \bar{n} e^{-q\phi/kT} .$$

# density for $\phi = 0$

... derived from Debye-Hückel theory (see e.g. Jackson 1975)

- Typically $|q\phi| \ll kT$ ⟶ approximation:

$$n_i = \bar{n}_i\left(1 - \frac{Z_i e\phi}{kT}\right) , \quad n_e = \bar{n}_e\left(1 + \frac{e\phi}{kT}\right) ,$$

Which shows directly the decrease of ion density and increase of e⁻ density.

## Nuclear energy production
### Electron screening

- total (electrical) charge density $\sigma$ for all types of ions ($n_i$)

For $\phi = 0$ (neutral gas) $\bar{\sigma} = 0$ i.e.: $\quad \bar{\sigma} = \sum_i (Z_i e)\bar{n}_i - e\bar{n}_e = 0$ ,

$\phi \neq 0$ : $\quad \sigma = \sum_i (Z_i e) n_i - e n_e$

$$= \sum_i -\frac{(Z_i e)^2\phi}{kT}\bar{n}_i - \frac{e^2\phi}{kT}\bar{n}_e .$$

- Combine last two terms to obtain

$$\boxed{\sigma = -\chi\frac{e^2\phi}{kT} n ,}$$

where total particle density $n$, and average value $\chi$ are

$$n = n_e + \textstyle\sum_i n_i , \qquad \chi := \frac{1}{n}\left(\sum_i Z_i^2 \bar{n}_i + \bar{n}_e\right) = \mu \sum_i \frac{Z_i(Z_i+1)}{A_i} X_i .$$

## Nuclear energy production
### Electron screening

- $\sigma$ and $\phi$ are also connected via the 'Poisson equation'

$$\nabla^2\phi = -4\pi\sigma\ .$$

which has the solution (for a point charge $Ze$) and for spherical geometry

$$\boxed{\phi = \frac{Ze}{r}\mathrm{e}^{-r/r_\mathrm{D}}} \quad \text{with} \quad r_\mathrm{D} = \left(\frac{kT}{4\pi\chi e^2 n}\right)^{1/2}$$

……… being the Debye-radius (to some sense the $r$ of $e^-$ cloud; $r_D$=$10^{-8}$…$10^{-9}$ cm.)

For 'normal stars' $\frac{r}{r_D} \simeq \frac{r_0}{r_\mathrm{D}} << 1$ ; $r_0 = Z_1Z_2e^2/E_0 \simeq 10^{-11}$ cm

classical particle distant at Gamov peak

$$\longrightarrow \boxed{\phi \simeq \frac{Ze}{r}\left(1-\frac{r}{r_\mathrm{D}}\right) = \underbrace{\frac{Ze}{r}}_{\text{unshielded potential}} - \frac{Ze}{r_\mathrm{D}}}$$

## Nuclear energy production
### Electron screening

- Accordingly the Coulomb barrier $E_{CB}$ is reduced by Debye energy $E_D$

$$E_\mathrm{CB} = \boxed{\frac{Z_1Z_2e^2}{r}} - \underbrace{\boxed{\frac{Z_1Z_2e^2}{r_\mathrm{D}}}}_{\text{Debye energy } E_D}\ .$$

Thereby increasing the probability with which particles can tunnel through the Coulomb barrier, leading to an increase of

$$\langle\sigma v\rangle_\mathrm{screen} \propto S(E_0)\int_0^\infty \underbrace{\mathrm{e}^{-bE^{-1/2}}}_{\text{Coulomb barrier}}\ \underbrace{\mathrm{e}^{-E/kT}}_{\text{M-B tail of thermal energy}}\ \underbrace{\mathrm{e}^{E_\mathrm{D}/kT}}_{\text{screening factor "f"}}\,\mathrm{d}E\ .$$

Increase by "f": $$\langle\sigma v\rangle_\mathrm{screen} = \langle\sigma v\rangle\ \underbrace{\mathrm{e}^{E_\mathrm{D}/kT}}_{\text{"f"}}$$

## Nuclear energy production
### Electron screening

$$E_\mathrm{D} = \frac{Z_1Z_2e^2}{r_\mathrm{D}}$$

$$r_\mathrm{D} = \left(\frac{kT}{4\pi\chi e^2 n}\right)^{1/2}$$

$$\chi \propto \rho$$

- weak screening: $\frac{E_\mathrm{D}}{kT} \ll 1 \rightarrow \frac{E_\mathrm{D}}{kT} \propto \left(\frac{\rho}{T^3}\right)^{1/2}$

**For solar case:** $\langle\sigma v\rangle_\mathrm{screen} \simeq 1.1 \times \langle\sigma v\rangle$

- strong screening: $\frac{E_\mathrm{D}}{kT} \gg 1 \rightarrow \frac{E_\mathrm{D}}{kT} \propto \frac{\rho^{1/3}}{T}$

Screening factor depends strongly on $\rho$ & $T$

## Nuclear energy production
### Hydrogen burning

**(a) pp chain:** $4\,{}^1\mathrm{H} \rightarrow {}^4\mathrm{He} + 2\mathrm{e}^+ + 2\nu_\mathrm{e}$

maintain charge balance (conserve lepton number)

no 4-body reactions → sub-reactions

$${}^1\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^2\mathrm{H} + e^+ + \nu_e$$ (controlled by weak interaction) → $14 \times 10^9$ y

$${}^2\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^3\mathrm{He} + \gamma$$ 6 s

2x

$T$ ~ 15 mill. K

$10^6$ y $\quad {}^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + 2\,{}^1\mathrm{H}$

(pp1)

$$Q_\mathrm{pp1} = 26.73 - 2\times 0.263\ \mathrm{MeV}$$

## Nuclear energy production

### Hydrogen burning

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$^2\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^3\mathrm{He} + \gamma$ — 6 s

$T \sim 15$ mill. K

2x

$10^6$ y — $^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + 2\,^1\mathrm{H}$ (pp1)

$Q_{\mathrm{pp1}} = 26.73 - 2 \times 0.263\,\mathrm{MeV}$

$^3\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^7\mathrm{Be} + \gamma$

$^7\mathrm{Be} + e^- \rightarrow {}^7\mathrm{Li} + \nu_e$

$^7\mathrm{Li} + {}^1\mathrm{H} \rightarrow {}^4\mathrm{He} + {}^4\mathrm{He}$ (pp2)

$Q_{\mathrm{pp2}} = 26.47 - 0.80\,\mathrm{MeV}$

## Nuclear energy production

### Hydrogen burning

**(a) pp chain:** $4\,^1\mathrm{H} \rightarrow {}^4\mathrm{He} + 2\mathrm{e}^+ + 2\nu_e$ — maintain charge balance (conserve lepton number); no 4-body reactions → sub-reactions

$^1\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^2\mathrm{H} + e^+ + \nu_e$ (controlled by weak interaction) → $14 \times 10^9$ y

$^2\mathrm{H} + {}^1\mathrm{H} \rightarrow {}^3\mathrm{He} + \gamma$ — 6 s

$T \sim 15$ mill. K

2x

$10^6$ y — $^3\mathrm{He} + {}^3\mathrm{He} \rightarrow {}^4\mathrm{He} + 2\,^1\mathrm{H}$ (pp1)

$Q_{\mathrm{pp1}} = 26.73 - 2 \times 0.263\,\mathrm{MeV}$

$^3\mathrm{He} + {}^4\mathrm{He} \rightarrow {}^7\mathrm{Be} + \gamma$

$^7\mathrm{Be} + e^- \rightarrow {}^7\mathrm{Li} + \nu_e$

$^7\mathrm{Li} + {}^1\mathrm{H} \rightarrow {}^4\mathrm{He} + {}^4\mathrm{He}$ (pp2)

$Q_{\mathrm{pp2}} = 26.47 - 0.80\,\mathrm{MeV}$

$^7\mathrm{Be} + {}^1\mathrm{H} \rightarrow {}^8\mathrm{B} + \gamma$

$^8\mathrm{B} \rightarrow {}^8\mathrm{Be} + e^+ + \nu_e$

$^8\mathrm{Be} \rightarrow {}^4\mathrm{He} + {}^4\mathrm{He}$ (pp3)

$Q_{\mathrm{pp3}} = 26.47 - 7.20\,\mathrm{MeV}$

## Nuclear energy production

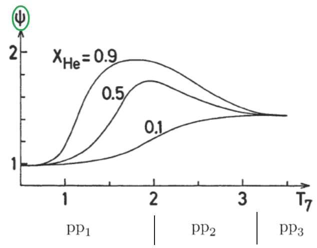
### Hydrogen burning

**(a) pp chain:** $4\,^1\mathrm{H} \rightarrow {}^4\mathrm{He} + 2\mathrm{e}^+ + 2\nu_e$ — no 4-body reactions → sub-reactions

shielding factor

$$\varepsilon_{pp} = 2.57 \times 10^4 \psi f_{11} g_{11} \varrho X_1^2 T_9^{-2/3} \mathrm{e}^{-3.381/T_9^{1/3}} \propto X^2 \rho T^n; \quad n \simeq 4 \; (T = 15 \times 10^6\,\mathrm{K})$$

$$g_{11} = (1 + 3.82 T_9 + 1.51 T_9^2 + 0.144 T_9^3 - 0.0114 T_9^4)$$




## Nuclear energy production

### Hydrogen burning

**(b) CNO cycle:** (C, N & O are catalysts) $Q_{\mathrm{CNO}} \simeq 26.73 - 1.71\,\mathrm{MeV}$

CNO (I)

| | |
|---|---|
| $^{12}\mathrm{C} + {}^1\mathrm{H} \rightarrow {}^{13}\mathrm{N} + \gamma$ | $1.3 \times 10^7$ y |
| $^{13}\mathrm{N} \rightarrow {}^{13}\mathrm{C} + e^+ + \nu_e$ (positron decay (p→n)) | 7 min |
| $^{13}\mathrm{C} + {}^1\mathrm{H} \rightarrow {}^{14}\mathrm{N} + \gamma$ | $2.7 \times 10^6$ y |
| $^{14}\mathrm{N} + {}^1\mathrm{H} \rightarrow {}^{15}\mathrm{O} + \gamma$ | $3.2 \times 10^8$ y |
| $^{15}\mathrm{O} \rightarrow {}^{15}\mathrm{N} + e^+ + \nu_e$ | 82 s |
| $^{15}\mathrm{N} + {}^1\mathrm{H} \rightarrow {}^{12}\mathrm{C} + {}^4\mathrm{He}$ | $1.1 \times 10^5$ y |

$T \sim 20$ mill. K

## Nuclear energy production

### Hydrogen burning

**(b) CNO cycle**: (C, N & O are catalysts) $Q_{\rm CNO} \simeq 26.73 - 1.71\,{\rm MeV}$

CNO (I)

| | | | |
|---|---|---|---|
| $^{12}C + {}^{1}H \to {}^{13}N + \gamma$ | | $1.3 \times 10^{7}$ y |
| $^{13}N \to {}^{13}C + e^{+} + \nu_e$ | positron decay (p⟶n) | 7 min |
| $^{13}C + {}^{1}H \to {}^{14}N + \gamma$ | | $2.7 \times 10^{6}$ y |
| $^{14}N + {}^{1}H \to {}^{15}O + \gamma$ | | $3.2 \times 10^{8}$ y |
| $^{15}O \to {}^{15}N + e^{+} + \nu_e$ | | 82 s |
| $^{15}N + {}^{1}H \to {}^{12}C + {}^{4}He$ | | $1.1 \times 10^{5}$ y |

CNO(II) [$10^3$ times less likely]

$\to {}^{16}O + \gamma$

$^{16}O + {}^{1}H \to {}^{17}F + \gamma$

$^{17}F \to {}^{17}O + e^{+} + \nu_e$

$^{17}O + {}^{1}H \to {}^{14}N + {}^{4}He$

$T \sim 20$ mill. K

## Nuclear energy production

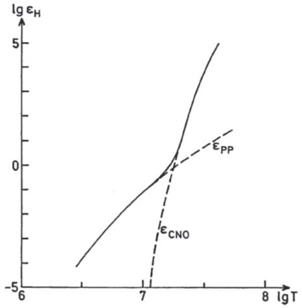
### Hydrogen burning

**(b) CNO cycle:**

$$\varepsilon_{\rm CNO} = 8.24 \times 10^{25} g_{14,1} \underbrace{X_{\rm CNO}}_{X_C+X_N+X_O} X_1 \rho T_9^{-2/3} e^{(-15.231 T_9^{-1/3} - (T_9/0.8)^2)}$$

$$\propto XZ\rho T^{n}; \quad n \simeq 20 \; (T = 15\times10^6\,{\rm K})$$

$$g_{14,1} = (1 - 2.00T_9 + 3.41T_9^2 - 2.43T_9^3)\,,$$




## Nuclear energy production

### Helium burning

**Triple α reaction**: $3\,{}^4{\rm He} \to {}^{12}{\rm C} + \gamma$ $(Q_{3\alpha} = 7.275\,{\rm MeV})$

performed in 2 steps: $^4{\rm He} + {}^4{\rm He} \rightleftarrows {}^8{\rm Be}\,,$ (decays after $10^{-16}$ s)

$^8{\rm Be} + {}^4{\rm He} \to {}^{12}{\rm C} + \gamma\,.$

$$\varepsilon_{3\alpha} = 5.09 \times 10^{11} f_{3\alpha} \rho^2 X_4^3 T_8^{-3} e^{-44.027/T_8} \quad \propto Y^3 \rho^2 T^{30} \quad \boxed{T = 1 - 2\times10^8\,{\rm K}}$$

Once sufficient $^{12}$C is produced, additional to *triple α reaction*, **successive α-caption**:

$$^{12}{\rm C} + {}^4{\rm He} \to {}^{16}{\rm O} + \gamma\,, \quad (Q = 7.162\,{\rm MeV})$$
$$^{16}{\rm O} + {}^4{\rm He} \to {}^{20}{\rm Ne} + \gamma\,,$$
$$^{20}{\rm Ne} + {}^4{\rm He} \to {}^{24}{\rm Mg} + \gamma\,,$$
$$\vdots \qquad ;$$

Also: $^{14}{\rm N}({}^4{\rm He}, e^+\nu_e)\,{}^{18}{\rm O}({}^4{\rm He},\gamma)\,{}^{22}{\rm Ne}\,.$

## Nuclear energy production

### Carbon burning

When He is exhausted, next element to react is $^{12}$C $(T_8 \simeq 5...10)$. $(Q \simeq 13\,{\rm MeV})$

| | $Q/1{\rm MeV}$ | |
|---|---|---|
| $^{12}{\rm C} + {}^{12}{\rm C} \to {}^{24}{\rm Mg} + \gamma$ , | 13.931 | |
| $\to {}^{23}{\rm Mg} + n$ , | −2.605 | (endothermic reaction) |
| $\to {}^{23}{\rm Na} + p$ , | 2.238 | most probable |
| $\to {}^{20}{\rm Ne} + \alpha$ , | 4.616 | most probable |
| $\to {}^{16}{\rm O} + 2\alpha$ , | −0.114 | |

$$\varepsilon_{\rm CC} \approx 1.86 \times 10^{43} f_{\rm CC} \varrho X_{12}^2 T_9^{-3/2} T_{9a}^{5/6}$$
$$\times \exp[-84.165/T_{9a}^{1/3} - 2.12\times10^{-3} T_9^3]$$

with $T_{9a} = T_9/(1 + 0.0396 T_9)$.

Screening factor $f_{\rm CC}$ may become important.

## Nuclear energy production

**Oxygen burning** $(Q \simeq 16\,\mathrm{MeV})$

Coulomb barrier already so high that **temperatures** $T_9 > 1$ **are required**.

$$\begin{aligned} {}^{16}\mathrm{O} + {}^{16}\mathrm{O} &\to {}^{32}\mathrm{S} + \gamma \ , \quad 16.541 \\ &\to {}^{31}\mathrm{P} + p \ , \quad 7.677 \\ &\to {}^{31}\mathrm{S} + n \ , \quad 1.453 \\ &\to {}^{28}\mathrm{Si} + \alpha \ , \quad 9.593 \\ &\to {}^{24}\mathrm{Mg} + 2\alpha \ , \quad -0.393 \end{aligned}$$

most frequent

2nd most frequent

***Photodisintegration*** **and subsequent α capture**:
($T$ so high that photons in MeV range; these may be absorbed by nuclei, causing them to break up, typically in an α decay), e.g. ***neon disintegration*** **(burning)**

$${}^{20}\mathrm{Ne} + \gamma \to {}^{16}\mathrm{O} + \alpha \ , \quad Q = -4.73\,\mathrm{MeV}$$

(endothermic occurs before oxygen burning)

α particle used in He burning: $\quad {}^{20}\mathrm{Ne} + {}^{4}\mathrm{He} \to {}^{24}\mathrm{Mg} + \gamma \ ,$

net reaction: $\quad 2\,{}^{20}\mathrm{Ne} + \gamma \to {}^{16}\mathrm{O} + {}^{24}\mathrm{Mg} + \gamma \ , \quad Q = +4.583\,\mathrm{MeV} \ .$

Silicon burning: two ${}^{28}\mathrm{Si} \to {}^{56}\mathrm{Fe}$ → end → supernova explosion.

## Nuclear energy production

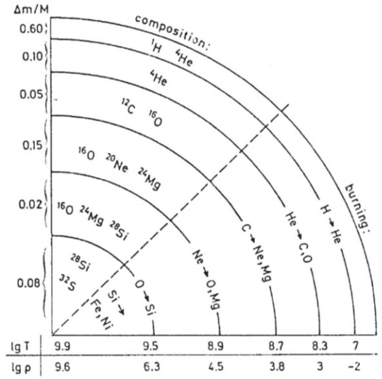



**Figure 14.3:** Schematic illustration (not to scale) of the "onion-shell" structure in the interior of a highly evolved massive star. Along the vertical radius and below the horizontal radius some typical values of the mass, the temperature (in K) and the density (in $\mathrm{g\,cm^{-3}}$) are indicated. (From Kippenhahn & Weigert 1990.)

## Nuclear energy production

**Neutrinos**

Have extremely small cross section $\sigma_\nu \simeq (E_\nu/m_e c^2)^2\, 10^{-44}\,\mathrm{cm}^2$ .

Neutrinos in MeV range have mean free path (μ=1)

$$\ell_\nu = \frac{1}{n\sigma_\nu} = \frac{\mu m_u}{\varrho \sigma_\nu} \approx \frac{2\times 10^{20}\mathrm{cm}}{\varrho}$$

Normal stellar matter : $\rho \simeq 1\,\mathrm{g\,cm^{-3}} \ : \ \ell_\nu \simeq 100\,\mathrm{parsec}$ .

$\rho \simeq 10^6\,\mathrm{g\,cm^{-3}} \ : \ \ell_\nu \simeq 300\,\mathrm{R}_\odot$ .

collapsing stellar core: $\rho \simeq 10^{14}\,\mathrm{g\,cm^{-3}} \ : \ \ell_\nu \simeq 20\,\mathrm{km}$ . ⟶ reabsorbed within star.

Also important for cooling cores of stars.

## Nuclear energy production

**Neutrinos**

Hydrogen burning (energy loss due to escaping neutrinos):

| | | |
|---|---|---|
| ${}^1\mathrm{H} + {}^1\mathrm{H} \to {}^2\mathrm{H} + e^+ + \nu$ | $(pp1,2,3)$ | $Q_\nu = 0.263\,\mathrm{MeV}$ |
| ${}^7\mathrm{Be} + e^- \to {}^7\mathrm{Li} + \nu$ | $(pp2)$ | $0.80\,\mathrm{MeV}$ |
| ${}^8\mathrm{B} \to {}^8\mathrm{Be} + e^+ + \nu$ | $(pp3)$ | $7.2\,\mathrm{MeV}$ |
| ${}^{13}\mathrm{N} \to {}^{13}\mathrm{C} + e^+ + \nu$ | (CNO) | $0.71\,\mathrm{MeV}$ |
| ${}^{15}\mathrm{O} \to {}^{15}\mathrm{N} + e^+ + \nu$ | (CNO) | $1.0\,\mathrm{MeV}$ |

With ~ 4x10$^{-5}$ erg/cycle Sun produces ~ 2x10$^{38}$ (electron) neutrinos every second!

⟶ Flux of ~ 10$^{11}$ solar neutrinos per cm$^2$ and second at Earth!

Urca process at extreme densities (no nuclear reactions necessary):

e$^-$ capture:

$$\underbrace{(Z,A)}_{\text{Nucleus with charge } Z \text{ and atomic weight } A} + \underbrace{e^-}_{\text{degenerate}} \to (Z-1,A) + \nu \ , \qquad {}^{35}\mathrm{Cl}\ (e^-,\nu)^{35}\mathrm{S}$$

$$(Z-1,A) \to (Z,A) + e^- + \bar{\nu} \ . \qquad {}^{35}\mathrm{S}\ (e^-\bar{\nu})^{35}\mathrm{Cl}$$

## Nuclear energy production

**Neutrinos**

other processes without nuclear reactions