The equation of state of stellar matter

Crystallization

- so far any interaction between ions were neglected (= ideal gas)
- not valid for high ρ and low T.

-if thermal kinetic energy kT becomes similar to electrostatic (potential) binding energy (Coulomb energy) ions tend to form a rigid lattice -> minimizes their total energy

Def.: coupling parameter $\Gamma_c = \dfrac{\text{potential (Coulomb) binding energy}}{\text{(thermal) kinetic energy}}$

$$\Gamma_c = \frac{(Ze)^2}{r_{\rm ion} kT} \simeq 2.7 \times 10^{-3} \frac{Z^2 \, n_{\rm ion}^{1/3}}{T}$$

$-Ze$ … ion charge

$r_{\rm ion}$ … mean separation between ions

$\Gamma_c << 1$ … ions have B-distribution

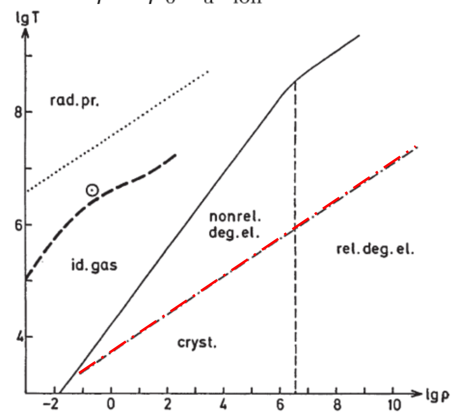
$\Gamma_c >> 1$ … ions try to form a crystal that has a lower energy

The equation of state of stellar matter

Crystallization

Critical value for transition (Shapiro & Teukolsky 1983): $\Gamma_c \simeq 170$.

with $\rho = \mu_0 m_u n_{\rm ion}$ we obtain estimate for critical (melting) temperature T_m:



$$T_m \approx \frac{Z^2 e^2}{\Gamma_c k} \left(\frac{4\pi\varrho}{3\mu_0 m_u} \right)^{1/3}$$

$$= 2.3 \times 10^3 Z^2 \mu_0^{-1/3} \varrho^{1/3}$$

Such conditions are found in cooling white dwarfs

The equation of state of stellar matter

Neutronization

high-energy e⁻ can combine with protons to form neutrons if total e⁻ energy is:

$$E_{\rm tot} > E^* = c^2(m_{\rm n} - m_{\rm p}) \ .$$

At relatively low ρ the neutron will decay within 11 min to produce proton-e⁻ pair with the e⁻ having energy $E^*_{\rm kin} = E^* - m_{\rm e}c^2$.

However, for **complete degeneracy** Fermi energy $E_{\rm F}$ could $> E^\star_{\rm kin}$ and released e⁻ have not enough energy to find empty cell in phase space → neutron cannot decay

Fermi sea of e- stabilizes neutrons if $E^*_{\rm kin} \leq E_{\rm F}$.

Using $p = \frac{1}{c}(E^2 - m_{\rm e}^2 c^4)^{1/2}$

and $E = E_{\rm kin} + m_{\rm e}c^2 = E_{\rm F} + m_{\rm e}c^2 = c^2(m_{\rm n} - m_{\rm p}) \simeq 1.29 \times 10^6\,{\rm eV}$ → $p_{\rm F}$

→ $x = p_{\rm F}/m_{\rm e}c \simeq 2.2$ & $n_{\rm e} = \rho/\mu_{\rm e}m_{\rm u} = 8\pi m_{\rm e}^3 c^3/3h^3 x^3$ & $\mu_{\rm e} = 2$

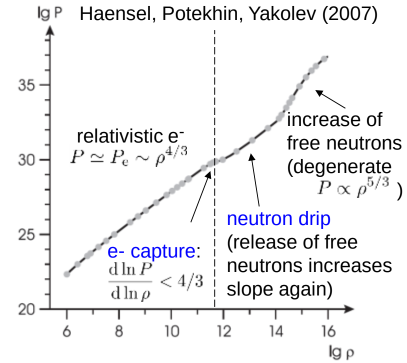
→ $\rho_{\rm crit} \simeq 2.4 \times 10^7\,{\rm gcm}^{-3}$ i.e. for $\rho > \rho_{\rm crit}$ proton-e⁻ gas → neutron gas.

The equation of state of stellar matter

Neutronization

In stars situation is more complicated: at high ρ, plasma contains heavy nuclei, which capture e- ("inverse β decay") to become neutron-rich isotopes → e- energy needs to be higher than $E_{\rm F}$.

If nuclei become too n-rich, they break up & release n(s) → neutron drip.



The equation of state of stellar matter

A self-consistent approximate approach

Idea: find a single expression for the EOS from which all thermodynamic quantities e.g., ρ, U, c_p, δ, etc, are consistently derived for given P, T and X_i

Ansatz:
use TD potential of free energy $F(T,V,\{N_i\}) = U - TS$ and find reaction equilibrium by selecting those $\{N\}$ that **minimizes** F (maximizes entropy S) for given T, V, subject to condition that total numbers of free e^- and any nuclei are constant.

From minimized free energy $F(T,V,\{N_i\})$ all TD quantities can be derived, e.g.

$$P = -\left(\frac{\partial F}{\partial V}\right)_T \,, \quad S = -\left(\frac{\partial F}{\partial T}\right)_V \,, \quad U = -T^2\left(\frac{\partial}{\partial T}\frac{F}{T}\right)_V \,.$$

$$c_p = -\frac{T}{\rho}\left(\frac{\partial^2 F}{\partial T^2}\right)_V \,.$$

The equation of state of stellar matter

A self-consistent approximate approach

Start from canonical partition function (Zustandssumme) Z .

Consider physical system (with Hamiltonian H) confined in a box of volume V in contact with a heat reservoir at temperature T:

$Z = \mathrm{Tr}\,(\exp[-H/kT])$... sum over all diagonal terms of Hamilton operator, which includes the sum over all internal excitation states j

e.g., $Z_{\mathrm{int}}^{(i)} = \sum_{j=0}^{\infty} g_{ij} \exp(-E_{ij}/kT)$ of species i ; $Z_{\mathrm{int}} = \prod_i Z_{\mathrm{int}}^{(i)}$

The free energy $F(T,V,\{Ni\})$ is then obtained from:

$$\boxed{F(T,V,\{N\}) \;=\; -kT\ln(Z)}$$

Statistical mechanics - thermodynamics

$$S = k \ln W \quad = -k \sum p_i \ln p_i$$

Ludwig Boltzmann
(1844 - 1906)

Probability

$$p_i = \frac{1}{Z} \exp\left(-E_i/kT\right)$$

Partition function (canonical)

$$Z = \sum \exp\left(-E_i/kT\right)$$

Helmholtz free energy *F*

$$F = -kT \ln Z$$

The equation of state of stellar matter

A self-consistent approximate approach

Partition function: $$Z \;=\; Z_{\rm e}\, Z_{\rm trans}\, Z_{\rm int}\, Z_{\rm rad}\, Z_{\rm conf}$$

$$\downarrow$$

Free energy: $$F \;=\; F_{\rm e} + F_{\rm trans} + F_{\rm int} + F_{\rm rad} + F_{\rm conf}$$

- $F_{\rm e}$: contribution from free electrons (including effects of degeneracy, as appropriate)
- $F_{\rm trans}$: contribution from the motion of heavy particles
- $F_{\rm int}$: contribution from the internal states in atoms and ions
- $F_{\rm rad}$: contribution from radiation
- $F_{\rm conf}$: the 'configuration' contribution, resulting from the finite size of atoms and ions, and the Coulomb interaction.

The equation of state of stellar matter

A self-consistent approximate approach

Saha equation can be derived from minimizing free energy $F(T,V,\{Ni\})$ (e.g. Däppen & Guzik (2000)).

Additional 'corrections', such as the electron chemical potential, $\Delta\mu$, can than easily and consistently be added to F by ΔF .

$$\frac{n_{r+1}}{n_r} n_\mathrm{e} = \frac{u_{r+1}}{u_r} 2 \frac{(2\pi m_\mathrm{e} k_\mathrm{B} T)^{3/2}}{h^3} \mathrm{e}^{-\chi_r/k_\mathrm{B}T + \Delta\mu}$$

$$\Delta\mu = -\frac{1}{k_\mathrm{B}T} \left(\frac{\partial \Delta F}{\partial n_\mathrm{e}} \right)_{T,V} .$$

The equation of state of stellar matter

A self-consistent approximate approach

Tackling the problem of the divergent partition function Z_int

$$F_\mathrm{int} = -k_\mathrm{B} T \sum_k \sum_j \ln \left[\sum_i w_{ijk} g_{ijk} \exp(-E_{ijk}/k_\mathrm{B}T) \right] .$$

k …. nr. of elements

j …. nr. of ionization states of each element

i …. nr. of bound (energy) states of each element

w_{ijk} …. newly introduced weights describing probability that state exists (MHD EOS; Mihalas, Hummer, Däppen 1988)

The equation of state of stellar matter

A self-consistent approximate approach

$$F_{\rm conf} = F_{\rm FV} \ + \ F_{\rm DH} \ .$$

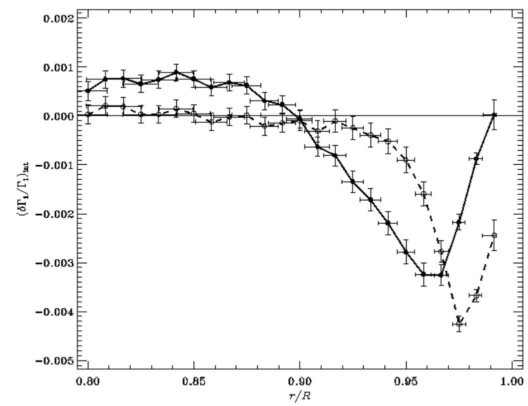
$F_{\rm FV}$ …. Finite volume of atoms and ions → "pressure (density) ionization"

$F_{\rm DH}$ …. Debye-Hückel approximation for Coloumb effects
(screening effect through electrostatic potential of ions)

$$F_{\rm DH} = -\frac{k_{\rm B} T V}{12 \pi r_{\rm D}^3} \ ,$$

$$r_{\rm D} \simeq \left[\frac{k_{\rm B} T}{4\pi (n_{\rm e} e^2 + n_{\rm i} e^2)} \right]^{1/2}$$ …. Debye length

Testing solar thermodynamics



• MHD

○ OPAL

Basu, Däppen & Nayfonov (1999; ApJ 518, 985)