Equation Of State (EOS)

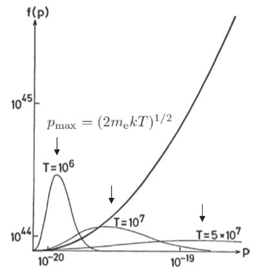
Degenerate Electron Gas

Degenerate Electron Gas

Boltzmann (B-statistics) distribution (classical picture)

- we consider gas of high pressure with dV being fully pressure ionized.
- number dN_e of free electrons in dV and spherical shell $[p,\ p+\mathrm{d}p]$ is:

$$f(p)\mathrm{d}p\mathrm{d}V = n_\mathrm{e}\frac{4\pi p^2}{(2\pi m_\mathrm{e}kT)^{3/2}}\exp\left(-\frac{p^2}{2m_\mathrm{e}kT}\right)\mathrm{d}p\mathrm{d}V\ , \qquad (p^2=p_x^2+p_y^2+p_z^2);$$



$T\downarrow \rightarrow p_{max}$ to smaller p values for

$$n_\mathrm{e}\mathrm{d}V = \mathrm{d}V\int_0^\infty f(p)\mathrm{d}p \quad = \text{constant.}$$

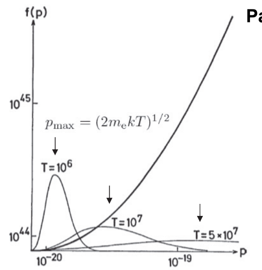
- electrons are fermions (spin ½).
- B-statistics **violates Pauli's exclusion principle.**

Degenerate Electron Gas

W. Pauli: each quantum cell (dp_x dp_y dp_z dx dy dz) can hold only 2 electrons.

quantum cell volume: dp_xdp_ydp_zdV = h^3 .

number of electrons in shell $[p,\ p+\mathrm{d}p]$: $8\pi p^2\mathrm{d}p\mathrm{d}V/h^3$.



Pauli: $f(p)\mathrm{d}p\mathrm{d}V \leq \frac{8\pi p^2\mathrm{d}p\mathrm{d}V}{h^3}$.

$T\downarrow \rightarrow p_{max}$ to smaller p values for

$$n_\mathrm{e}\mathrm{d}V = \mathrm{d}V\int_0^\infty f(p)\mathrm{d}p \quad = \text{constant.}$$

Violation also for T = constant and high densities, since $f(p)$dp ~ n_e .

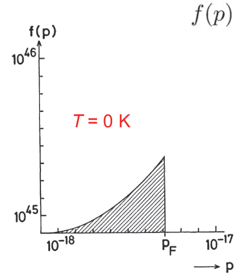
need to include quantum effects if either T too low or electron density too high, i.e. if electrons become *degenerate.*

Degenerate Electron Gas

The **completely degenerate electron gas @ T=0**

All electrons have lowest energy without violating Pauli's principle, i.e. all phase cells up to p_F are allocated by 2 electrons, all other cells above p_F are empty.

$$f(p) = \begin{cases} \frac{8\pi p^2}{h^3} & \text{for } p \leq p_\mathrm{F} \\ 0 & \text{for } p > p_\mathrm{F} \end{cases}$$



total number N_e of electrons:

$$n_\mathrm{e}\mathrm{d}V = \mathrm{d}V\int_0^{p_\mathrm{F}}\frac{8\pi p^2}{h^3}\mathrm{d}p = \frac{8\pi}{3h^3}p_\mathrm{F}^3\mathrm{d}V$$

$$p_\mathrm{F}\sim n_\mathrm{e}^{1/3}\ ;E_\mathrm{F}=p_\mathrm{F}^2/2m_e\sim n_\mathrm{e}^{2/3}$$

(non-relativistic)

Degenerate Electron Gas

The **completely degenerate electron gas @ *T*=0**

non-relativistic: $E_F = p_F^2/2m_e \sim n_e^{2/3}$ $\qquad p_F \sim n_e^{1/3}$

If n_e is sufficiently large p_F can become so high that electron *v* ~ speed of light *c*

⟶ **relativistic** (Landau & Lifschitz vol.2):

$$p = \frac{m_e v}{\sqrt{1 - v^2/c^2}} \, , \qquad \frac{v^2}{c^2} = \frac{p^2/(c^2 m_e^2)}{1 + p^2/(c^2 m_e^2)}$$

kin. energ. $E = m_e c^2 \left[\left(1 + \frac{p^2}{m_e^2 c^2} \right)^{1/2} - 1 \right] = E_{tot} - m_e c^2$

↑ rest energy (rest mass)

$$E_{tot} = m_e c^2 / \sqrt{1 - v^2/c^2}$$

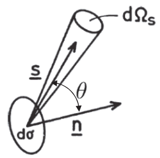
Degenerate Electron Gas

The **completely degenerate electron gas @ *T*=0**

Pressure = flux of momentum (through unit surface and second).

momentum flux in direction ***n*** of e^- moving into solid-angle element $d\Omega_s$:

$$f(p)v(p)p\mathrm{d}p \cos^2 \theta \mathrm{d}\Omega_s / 4\pi \; ;$$



total flux in direction ***n*** by integration over all *s* of a hemisphere:

$$P_e = \frac{1}{3} \int_0^\infty f(p) v(p) p \mathrm{d}p \quad ;$$

internal energy of electron gas per unit volume:

$$U_e = \int_0^\infty f(p) E(p) \mathrm{d}p \quad ;$$

$$n_e \mathrm{d}V = \mathrm{d}V \int_0^\infty f(p) \mathrm{d}p \quad .$$

Degenerate Electron Gas

The **completely degenerate electron gas @ *T*=0**

$$P_e = \frac{1}{3} \int_0^\infty f(p) v(p) p \mathrm{d}p \;=\; \frac{8\pi}{3h^3} \int_0^{p_F} v(p) p^3 \mathrm{d}p$$

with $p = \frac{m_e v}{\sqrt{1 - v^2/c^2}}$; $\xi \equiv \frac{p}{m_e c}$ and $\boxed{x \equiv \frac{p_F}{m_e c}}$ ..... importance of relativistic effects

$$P_e = \frac{8\pi c^5 m_e^4}{3h^3} \int_0^x \frac{\xi^4}{(1+\xi^2)^{1/2}} \mathrm{d}\xi \;=\; \frac{\pi c^5 m_e^4}{3h^3} f(x)$$

$$f(x) = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3 \ln \left[x + (1 + x^2)^{1/2} \right]$$

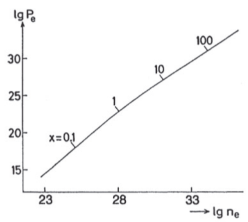
Degenerate Electron Gas

The **completely degenerate electron gas @ *T*=0**

Pressure of electron gas:

$$P_e = \frac{\pi c^5 m_e^4}{3h^3} f(x)$$

does not depend on *T*



Total number of electrons:

$$n_e \mathrm{d}V = \mathrm{d}V \int_0^{p_F} \frac{8\pi p^2}{h^3} \mathrm{d}p = \frac{8\pi}{3h^3} p_F^3 \mathrm{d}V$$

$$n_e = \frac{\rho}{\mu_e m_u} = \frac{8\pi m_e^3 c^3}{3h^3} x^3$$

$$\longrightarrow P_e = P_e(n_e; x) \; ;$$

Note: $\rho = \mu_0 m_u n \;=\; \mu_e m_u n_e$

$x \equiv \frac{p_F}{m_e c}$ ..... relativity parameter

Degenerate Electron Gas

The **completely degenerate electron gas @ *T*=0**

Non-relativistic limit (asymptotic behaviour) : $\frac{v_F^2}{c^2} = \frac{x^2}{1+x^2}$,

$x \to 0$: $f(x) \to \frac{8}{5}x^5 , \quad g(x) \to \frac{12}{5}x^5 ,$

$$P_e = \frac{8\pi m_e^4 c^5}{15h^3} x^5 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e m_u^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3}$$

$$P_e = 1.0036 \times 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} \text{(cgs)}$$

Independent of *T*

$$U_e = \frac{3}{2}P_e \ .$$

Degenerate Electron Gas

The **completely degenerate electron gas @ *T*=0**

Extreme relativistic limit:

$x \to \infty$: $f(x) \to 2x^4 , \quad g(x) \to 6x^4 ,$

$$P_e = \frac{2\pi m_e^4 c^5}{3h^3} x^4 = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8m_u^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{4/3}$$

$$P_e = 1.2435 \times 10^{15} \left(\frac{\rho}{\mu_e}\right)^{4/3} \text{(cgs)}$$

Independent of *T*

$$U_e = 3P_e \ .$$

Degenerate Electron Gas

Partial **degeneracy of electron gas**

For finite *T* not all electrons are densely packed in momentum space.
For high *T* we expect them to have a Boltzmann distribution.
We further expect a smooth transition from completely- to non-degenerate case.

Most probable occupation in momentum space described by **Fermi-Dirac** (F-D) statistics:

max. allowed occupation in shell $[p,\ p+\mathrm{d}p]$

$$f(p)\mathrm{d}p\mathrm{d}V = \frac{8\pi p^2 \mathrm{d}p\mathrm{d}V}{h^3} \frac{1}{1+\exp[E/(kT)-\psi]}$$

filling factor <1: fraction of occupied cells

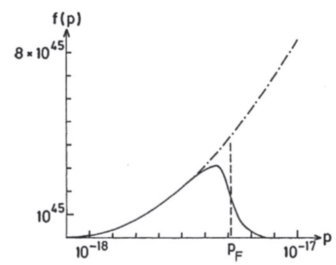
ψ …. degeneracy parameter (determines degree of partial degeneracy)

Degenerate Electron Gas

Partial **degeneracy of electron gas**

Fermi-Dirac (F-D) distribution function f(p) for partially degenerated electron gas:

$$f(p)\mathrm{d}p\mathrm{d}V = \frac{8\pi p^2 \mathrm{d}p\mathrm{d}V}{h^3} \frac{1}{1+\exp[E/(kT)-\psi]}$$



T = 1.9 x 10^7 K

ψ = 10

n_e = 10^{28} cm^{-3}

Degenerate Electron Gas

Partial degeneracy of electron gas

F-D: $f(p)\mathrm{d}p\mathrm{d}V = \frac{8\pi p^2 \mathrm{d}p\mathrm{d}V}{h^3} \frac{1}{1+\exp[E/(kT)-\psi]}$

Non-relativistic (e- density n_e): $(E = p^2/2m_e)$

$$n_e = \int_0^\infty f(p)\mathrm{d}p = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\mathrm{d}p}{1+\exp[p^2/(2m_e kT)-\psi]} = \frac{8\pi}{h^3}(2m_e kT)^{3/2} a(\psi)$$

where $a(\psi) = \int_0^\infty \frac{\eta^2}{1+\exp(\eta^2-\psi)}\mathrm{d}\eta \quad ; \ \eta \equiv p/(2m_e kT)^{1/2}$

$\longrightarrow \quad \psi = \psi\left(\frac{n_e}{T^{3/2}}\right)$ …. degeneracy parameter.

Degenerate Electron Gas

Partial degeneracy of electron gas

F-D: $f(p)\mathrm{d}p\mathrm{d}V = \frac{8\pi p^2 \mathrm{d}p\mathrm{d}V}{h^3} \frac{1}{1+\exp[E/(kT)-\psi]}$

Non-relativistic (e- density n_e): $(E = p^2/2m_e)$

$$n_e = \int_0^\infty f(p)\mathrm{d}p = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\mathrm{d}p}{1+\exp[p^2/(2m_e kT)-\psi]}$$

using $m_e\mathrm{d}E = p\mathrm{d}p$ & $p = (2m_e E)^{1/2}$

$$n_e = \frac{\rho}{\mu_e m_u} = \frac{4\pi}{h^3}(2m_e)^{3/2}\int_0^\infty \frac{E^{1/2}\mathrm{d}E}{1+\exp[E/(kT)-\psi]} = \frac{4\pi}{h^3}(2m_e kT)^{3/2} F_{1/2}(\psi)$$

Fermi-Dirac integrals: $F_\nu(\psi) = \int_0^\infty \frac{y^\nu}{1+\exp(y-\psi)}\mathrm{d}y \quad ; \ y = E/kT$

Degenerate Electron Gas

Partial degeneracy of electron gas

Table 15.1. Numerical values for Fermi–Dirac functions $F_{1/2}, F_{3/2}, F_2, and F_3$ (after Gong et al. 2001, using the computer program for numerical integration provided there).

Ψ	$F_{3/2}(\Psi)$	$F_{1/2}(\Psi)$	$F_2(\Psi)$	$F_3(\Psi)$
-4.00	0.024269	0.016128	0.036548	0.109768
-3.50	0.039931	0.026481	0.060169	0.180844
-3.00	0.065612	0.043366	0.098963	0.297802
-2.50	0.107581	0.070724	0.162525	0.490023
-2.00	0.175801	0.114588	0.266265	0.805319
-1.50	0.285772	0.183802	0.434567	1.320880
-1.00	0.460849	0.290501	0.705130	2.159840
-0.50	0.734659	0.449793	1.134368	3.515199
0.00	1.152804	0.678094	1.803085	5.682197
0.50	1.772794	0.990209	2.820969	9.098521
1.00	2.661683	1.396375	4.328331	14.389356
1.50	3.891976	1.900833	6.494369	22.412444
2.00	5.537254	2.502458	9.512668	34.298283
2.50	7.668804	3.196599	13.595529	51.482510
3.00	10.353715	3.976985	18.968568	75.729812
3.50				
4.00				
⋮				⋮
18.00	560.305110	51.106078	2003.217626	27854.240307
18.50	599.432825	53.239389	2171.404227	30984.066072
19.00	640.171486	55.401871	2348.840828	34373.077988
19.50	682.542825	57.593132	2535.777429	38035.338556
20.00	726.568284	59.812795	2732.464029	41985.285274

$$F_\nu(\psi) = \int_0^\infty \frac{y^\nu}{1+\exp(y-\psi)}\mathrm{d}y$$

Degenerate Electron Gas

Partial degeneracy of electron gas

Approximations to F-D integrals: $F_\nu(\psi) = \int_0^\infty \frac{y^\nu}{1+\exp(y-\psi)}\mathrm{d}y$

$\psi \rightarrow +\infty$: (strong degeneracy)

$$F_\nu(\psi) = \frac{\psi^{\nu+1}}{\nu+1}\{1 + 2[c_2(\nu+1)\nu\psi^{-2} + c_4(\nu+1)\nu(\nu-1)(\nu-2)\psi^{-4} + \ldots]\},$$

with $c_2 = \pi^2/12, c_4 = 7\pi^4/720.$

$\psi \rightarrow -\infty$: (e^- behave almost like an **ideal gas**)

$$F_\nu(\psi) = \int_0^\infty \frac{y^\nu dy}{1+e^{(y-\psi)}} \approx e^\psi \int_0^\infty y^\nu e^{-y} dy$$

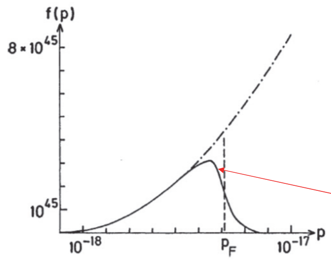
$\nu = 1/2 \quad \longrightarrow \quad F_{1/2}(\psi) \simeq \sqrt{\pi}\exp(\psi)/2$

Degenerate Electron Gas

Partial **degeneracy of electron gas**

Fermi-Dirac (F-D) distribution function f(p) for partially degenerated electron gas:

$$f(p)\mathrm{d}p\mathrm{d}V = \frac{8\pi p^2 \mathrm{d}p\mathrm{d}V}{h^3}\frac{1}{1+\exp[E/(kT)-\psi]}$$



T = 1.9 x 10[7] K

ψ = 10

n_e = 10[28] cm[-3]

the larger T the smoother, i.e. the Boltzmann-like the transition about p_F.

Degenerate Electron Gas

Partial **degeneracy of electron gas**

Non-relativistic (electron pressure): $(E = p^2/2m_\mathrm{e})$

$$P_\mathrm{e} = \frac{1}{3}\int_0^\infty f(p)v(p)p\mathrm{d}p \;=\; \frac{8\pi}{3h^3}\int_0^\infty p^3 v(p)\frac{\mathrm{d}p}{1+\exp(E/kT-\psi)}$$

with $p^3 v(p)\mathrm{d}p \;=\; m_\mathrm{e}^4 v^4 \mathrm{d}v \;=\; m_\mathrm{e}^3 v^3 \mathrm{d}E \;=\; m_\mathrm{e}^{3/2} 2^{3/2} E^{3/2}\mathrm{d}E$

$$P_\mathrm{e} \;=\; \frac{8\pi}{3h^3}(2m_\mathrm{e})^{3/2}\int_0^\infty \frac{E^{3/2}\mathrm{d}E}{1+\exp[E/(kT)-\psi]}$$

with $y = E/kT$

$$P_\mathrm{e} \;=\; \frac{8\pi}{3h^3}(2m_\mathrm{e}kT)^{3/2}kT F_{3/2}(\psi)$$

This equation together with equation for n_e form EOS to obtain $P_e(n_e, T)$.

Degenerate Electron Gas

Partial **degeneracy of electron gas**

EOS for **non-relativistic** ($x \to 0$) electron gas (**summary**):
$[p = (2m_\mathrm{e}E)^{1/2}]$

$$n_\mathrm{e} \;=\; \frac{\rho}{\mu_\mathrm{e} m_\mathrm{u}} \;=\; \frac{4\pi}{h^3}(2m_\mathrm{e}kT)^{3/2}F_{1/2}(\psi) \qquad (1)$$

$$P_\mathrm{e} \;=\; \frac{8\pi}{3h^3}(2m_\mathrm{e}kT)^{3/2}kT F_{3/2}(\psi) \qquad (2)$$

$$U_\mathrm{e} = \frac{3}{2}P_\mathrm{e}$$

For given T and n_e or ρ (1) provides ψ and with (2) P_e is obtained (and also U_e)

Degenerate Electron Gas

Partial **degeneracy of electron gas**

EOS for **extreme-relativistic** ($x \to \infty$) electron gas (summary):
$(p = E/c; \quad v = c)$

$$n_\mathrm{e} \;=\; \frac{\rho}{\mu_\mathrm{e} m_\mathrm{u}} \;=\; 8\pi\left(\frac{kT}{hc}\right)^3 F_2(\psi) \qquad (1)$$

$$P_\mathrm{e} \;=\; \frac{8\pi}{3h^3c^3}(kT)^4 F_3(\psi) \qquad (2)$$

$$U_\mathrm{e} = 3P_\mathrm{e}$$

For given T and n_e or ρ (1) provides ψ and with (2) P_e is obtained (and also U_e)

The equation of state of stellar matter

The ion gas (non-degenerate): $P_{\rm ion} = n_{\rm ion} kT = \frac{\mathcal{R}}{\mu_0}\rho T$ of same order as $P_{\rm e}$.

If ions of fermions type (e.g. protons, He[3]) they may become degenerate like electrons

⟶ use same equations as for electron gas with $m_{\rm e}$ replaced by $m_{\rm ion}$

i.e., eq. for n:
(non-relativist.)

$$\frac{n_j}{T^{3/2}} \propto m_j^{3/2} F_{1/2}(\psi) \qquad n_j = \{n_{\rm e}, n_{\rm ion}\} \quad m_j = \{m_{\rm e}, m_{\rm ion}\}$$

Suppose e- gas has certain $\psi^\star(n_{\rm e}^\star, T^\star)$
than the ion gas with same $T^\star$ will have the same $\psi^\star$ if:

$$n_{\rm ion} = \left(\frac{m_{\rm ion}}{m_{\rm e}}\right)^{3/2} n_{\rm e}^\star \simeq 8\times 10^4 n_{\rm e}^\star$$

because e- are degenerate (higher momentum) already at lower densities
⟶ $P = P_{\rm ion} + P_{\rm e} \simeq P_{\rm e}$

But: ions are main contributor to density $\rho = \mu_0 m_{\rm u} n_{\rm ion} = \mu_{\rm e} m_{\rm u} n_{\rm e}$

The equation of state of stellar matter

EOS for (normal stellar matter) and all degrees of degeneracy and relativistic effects:

$$P = P_{\rm ion} + P_{\rm e} + P_{\rm rad} = \frac{\Re}{\mu_0}\varrho T + \frac{8\pi}{3h^3}\int_0^\infty p^3 v(p)\frac{dp}{{\rm e}^{E/kT-\psi}+1} + \frac{a}{3}T^4 \qquad (1)$$

$$\rho = \frac{4\pi}{h^3}(2m_{\rm e})^{3/2} m_{\rm u}\mu_{\rm e}\int_0^\infty \frac{E^{1/2}{\rm d}E}{{\rm e}^{E/kT-\psi}+1} \qquad (2)$$

$$E = m_{\rm e}c^2\left[\left(1+\frac{p^2}{m_{\rm e}^2c^2}\right)^{1/2} - 1\right] \quad ; p = \frac{m_{\rm e}v}{\sqrt{1-v^2/c^2}}\ , \quad ; \ v(p) = \partial E/\partial p\ .$$

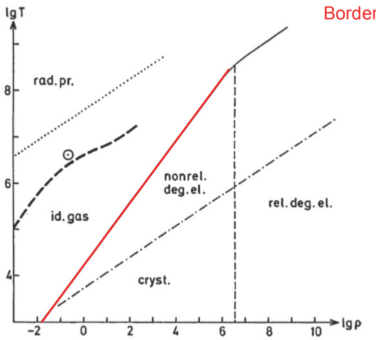
For given ρ , T and chemical composition μ_0 : from (2) → ψ → (1) → P

Internal energy u per unit mass:

$$u = \frac{U_{\rm ion}+U_{\rm e}+U_{\rm rad}}{\varrho} = \frac{3}{2}\frac{\Re}{\mu_0}T + \frac{8\pi}{h^3\varrho}\int_0^\infty \frac{p^2E(p)dp}{{\rm e}^{E/kT-\psi}+1} + \frac{aT^4}{\varrho}$$

The equation of state of stellar matter

where does degeneracy become important?



Border (ψ =const.): ideal – degenerate (non-rel.)

$$\frac{\Re}{\mu}\varrho T = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_{\rm e}}\left(\frac{\varrho}{\mu_{\rm e}m_{\rm u}}\right)^{5/3}$$

↓

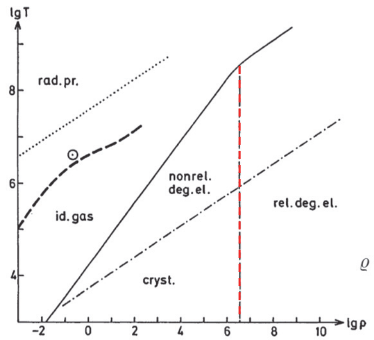
$$\frac{T}{\varrho^{2/3}} = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_{\rm e}\Re m_{\rm u}^{5/3}}\frac{\mu}{\mu_{\rm e}^{5/3}}$$

$$= 1.207\times 10^5 \frac{\mu}{\mu_{\rm e}^{5/3}}$$

compl. degen. (non-rel.) e-: $P_{\rm e} = \frac{8\pi m_{\rm e}^4c^5}{15h^3}x^5 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_{\rm e}m_{\rm u}^{5/3}}\left(\frac{\rho}{\mu_{\rm e}}\right)^{5/3}$

The equation of state of stellar matter

where do relativistic effects become important?



Border: non-relat. and relat. (degenerate)

rel. parameter $x \simeq 1$

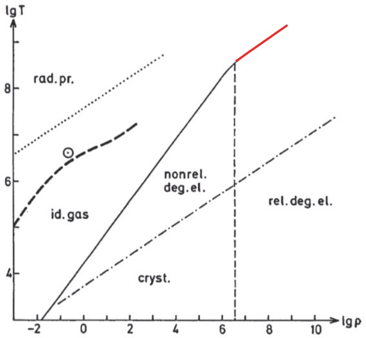
$$n_{\rm e}{\rm d}V = {\rm d}V\int_0^{p_{\rm F}}\frac{8\pi p^2}{h^3}{\rm d}p = \frac{8\pi}{3h^3}p_{\rm F}^3{\rm d}V$$

$$\rho = \mu_{\rm e}m_{\rm u}n_{\rm e} \quad \downarrow \quad x \equiv \frac{p_{\rm F}}{m_{\rm e}c}$$

$$\varrho = \frac{8\pi m_{\rm u}m_{\rm e}^3c^3}{3h^3}\mu_{\rm e} = 9.74\times 10^5\mu_{\rm e}\ ({\rm cgs})$$

$$\mu_{\rm e} = 2$$

The equation of state of stellar matter



Border: ideal – degenerate (relativistic)

$$\frac{\mathcal{R}}{\mu}\rho T = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8m_u^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{4/3}$$

$$\downarrow$$

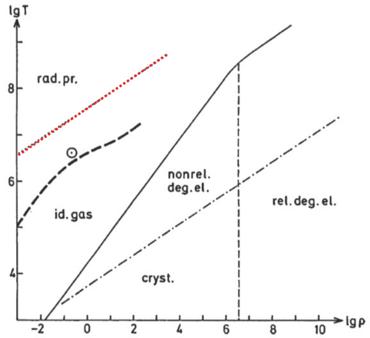
$$\frac{T}{\varrho^{1/3}} = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8\Re} \frac{1}{m_u^{4/3}} \frac{\mu}{\mu_e^{4/3}}$$

$$= 1.496 \times 10^7 \frac{\mu}{\mu_e^{4/3}}$$

compl. degen. (rel.) e-: $P_e = \frac{2\pi m_e^4 c^5}{3h^3} x^4 = \left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8m_u^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{4/3}$

The equation of state of stellar matter

where does radiation pressure become important?



Border: ideal gas and radiation pressure

$$\frac{\Re}{\mu}\varrho T = \frac{a}{3}T^4$$

$$\downarrow$$

$$\frac{T}{\varrho^{1/3}} = \left(\frac{3\Re}{a\mu}\right)^{1/3} = \frac{3.2 \times 10^7}{\mu^{1/3}}$$

The equation of state of stellar matter

Thermodynamic quantities, e.g. δ, c_p, ∇_{ad}, for some limiting cases (P_{rad}=0, ideal ions):

complete degeneracy & non-relativ.: $\alpha = 3/5,\ \delta = 0$

complete degeneracy & relativistic: $\alpha = 3/4,\ \delta = 0$

strong degeneracy $\psi >> 1$ & non-relativ.: $P_e \approx \frac{4}{15} B_1 (\psi kT)^{5/2}$, $B_1 = \frac{4\pi}{h^3}(2m_e)^{3/2}$

$$\varrho \approx \frac{2}{3}\mu_e m_u B_1 (\psi kT)^{3/2}$$

$$\eta := \frac{P_{ion}}{P_{ion} + P_e} \approx \frac{5}{2}\frac{\mu_e}{\mu_0}\frac{1}{\psi}$$

for small P_{ion} contribution: $P = P_e/(1-\eta) \approx (1+\eta)P_e$

using $\delta = -\left(\frac{\partial \ln \varrho}{\partial \ln T}\right)_P = -\left(\frac{\partial \ln \varrho}{\partial \ln T}\right)_\psi + \frac{\left(\frac{\partial \ln \varrho}{\partial \ln \psi}\right)_T \left(\frac{\partial \ln P}{\partial \ln T}\right)_\psi}{\left(\frac{\partial \ln P}{\partial \ln \psi}\right)_T} \approx \frac{3}{5}\eta \approx \frac{3}{2}\frac{\mu_e}{\mu_0}\frac{1}{\psi}$

The equation of state of stellar matter

Thermodynamic quantities, e.g. δ, c_p, ∇_{ad}, for some limiting cases (P_{rad}=0, ideal ions):

Extremely relativistic, $x \to \infty$,(and compl. degen. e-) gas:

$$\eta \approx 4\frac{\mu_e}{\mu_0}\frac{1}{\psi}\ , \quad \delta = \frac{3}{4}\eta = \frac{3\mu_e}{\mu_0}\frac{1}{\psi}\ .$$

$$P_e = \frac{B_2}{4}(\psi kT)^4\ , \quad B_2 = \frac{8\pi}{3c^3h^3}\ ,$$

$$\varrho = \mu_e m_u B_2 (\psi kT)^3$$

The equation of state of stellar matter

Thermodynamic quantities, e.g. δ, c_p, ∇_{ad}, for some limiting cases (P_{rad}=0, ideal ions):

Non-relativistic (degenerate e-): $U_e = 3P_e/2$ (non-deg. ions) $U_{ion} = 3P_{ion}/2$

$$\rightarrow u = \frac{U}{\varrho} = \frac{3}{2}\frac{P_{ion}+P_e}{\varrho} = \frac{3}{2}\frac{P}{\varrho}$$

independent of ψ!

$$c_P = \left(\frac{\partial u}{\partial T}\right)_P - \frac{P}{\varrho^2}\left(\frac{\partial \varrho}{\partial T}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P + \frac{P\delta}{\varrho T} = \frac{5}{2}\frac{P\delta}{\varrho T}$$

$$\nabla_{ad} = \frac{P\delta}{T\varrho c_P} = 2/5$$ (as in an ideal gas)

Extreme-relativistic (strong degen.): $U_e = 3P_e$ $U_{ion} = 3P_{ion}/2$

$$c_P = -\frac{4P}{\varrho^2}\left(\frac{\partial \varrho}{\partial T}\right)_P - \frac{3}{2}\frac{\Re}{\mu_0} = \frac{4P}{\varrho T}\delta - \frac{3}{2}\frac{\Re}{\mu_0}$$

$$\nabla_{ad} = 1/2 \qquad \delta = 3\frac{\mu_e}{\mu_0}\frac{1}{\psi}$$

The equation of state of stellar matter

Crystallization

- so far any interaction between ions were neglected (= ideal gas)
- not valid for high ρ and low T.

-if thermal kinetic energy *kT* becomes similar to electrostatic (potential) binding energy (Coulomb energy) ions tend to form a rigid lattice -> minimizes their total energy

Def.: coupling parameter Γ_c = potential (Coulomb) binding energy / (thermal) kinetic energy

$$\Gamma_c = \frac{(Ze)^2}{r_{ion}kT} \simeq 2.7\times 10^{-3}\frac{Z^2 n_{ion}^{1/3}}{T}$$

$\Gamma_c << 1$ ... ions have B-distribution

$\Gamma_c >> 1$ ... ions try to form a crystal that has a lower energy

$-Ze$ ... ion charge

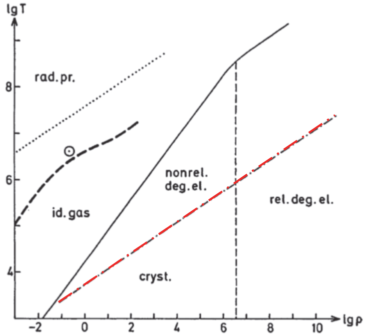
r_{ion} ... mean separation between ions

The equation of state of stellar matter

Crystallization

Critical value for transition (Shapiro & Teukolsky 1983): $\Gamma_c \simeq 170$.

with $\rho = \mu_0 m_u n_{ion}$ we obtain estimate for critical (melting) temperature T_m:



$$T_m \approx \frac{Z^2e^2}{\Gamma_c k}\left(\frac{4\pi\varrho}{3\mu_0 m_u}\right)^{1/3}$$

$$= 2.3\times 10^3 Z^2 \mu_0^{-1/3}\varrho^{1/3}$$

Such conditions are found in cooling white dwarfs

The equation of state of stellar matter

Neutronization

high-energy e⁻ can combine with protons to form neutrons if total e⁻ energy is:

$$E_{tot} > E^* = c^2(m_n - m_p) .$$

At relatively low ρ the neutron will decay within 11 min to produce proton-e⁻ pair with the e⁻ having energy $E^*_{kin} = E^* - m_e c^2$.

However, for **complete degeneracy** Fermi energy E_F could $> E^*_{kin}$ and released e⁻ have not enough energy to find empty cell in phase space → neutron cannot decay

Fermi sea of e- stabilizes neutrons if $E^*_{kin} \le E_F$.

Using $p = \frac{1}{c}(E^2 - m_e^2c^4)^{1/2}$

and $E = E_{kin} + m_e c^2 = E_F + m_e c^2 = c^2(m_n - m_p) \simeq 1.29\times 10^6\,\mathrm{eV}$ → p_F

→ $x = p_F/m_e c \simeq 2.2$ & $n_e = \rho/\mu_e m_u = 8\pi m_e^3 c^3/3h^3 x^3$ & $\mu_e = 2$

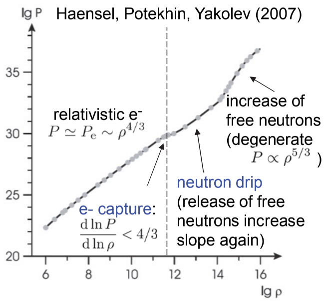
→ $\rho_{crit} \simeq 2.4\times 10^7\,\mathrm{gcm^{-3}}$ i.e. for $\rho > \rho_{crit}$ proton-e⁻ gas → neutron gas.

The equation of state of stellar matter

Neutronization

In stars situation is more complicated: at high ρ, plasma contains heavy nuclei, which capture e- ("inverse β decay") to become neutron-rich isotopes ⟶ e- energy needs to be higher than E_F.

If nuclei become too n-rich, they break up & release n(s) ⟶ neutron drip.



using $p = \frac{1}{c}(E^2 - m_e^2 c^4)^{1/2}$

and $E = E_{\rm kin} + m_e c^2 = E_F + m_e c^2 = c^2(m_n - m_p) =\simeq 1.29\times 10^6\, eV$

→ $x = p_F/m_e c \simeq 2.2$ & $n_e = \rho/\mu_e m_u = 8\pi m_e^3 c^3/3h^3 x^3$ & $\mu_e = 2$

→ $\rho_{\rm crit} \simeq 2.4\times 10^7\,{\rm gcm}^{-3}$ i.e. for $\rho > \rho_{\rm crit}$ proton-e⁻ gas → neutron gas.

The equation of state of stellar matter

A self-consistent approximate approach

Idea: find a single expression for the EOS from which all thermodynamic quantities e.g., ρ, U, c_p, δ, etc, are consistently derived for given P, T and X_i

Ansatz:
use TD potential of free energy $F(T,V,\{N_i\}) = U - TS$ and find reaction equilibrium by selecting those $\{N\}$ that **minimizes *F*** (maximizes entropy S) for given T,V, subject to condition that total numbers of free e^- and any nuclei are constant.

From minimized free energy $F(T,V,\{N_i\})$ all TD quantities can be derived, e.g.

$$P = -\left(\frac{\partial F}{\partial V}\right)_T \;,\quad S = -\left(\frac{\partial F}{\partial T}\right)_V \;,\quad U = -T^2\left(\frac{\partial}{\partial T}\frac{F}{T}\right)_V \;.$$

$$c_p = -\frac{T}{\rho}\left(\frac{\partial^2 F}{\partial T^2}\right)_V \;.$$

The equation of state of stellar matter

A self-consistent approximate approach

Start from canonical partition function (Zustandssumme) Z .

Consider physical system (with Hamiltonian H) confined in a box of volume V in contact with a heat reservoir at temperature T:

$Z = {\rm Tr}\left(\exp[-H/kT]\right)$... sum over all diagonal terms of Hamilton operator, which includes the sum over all internal excitation states j

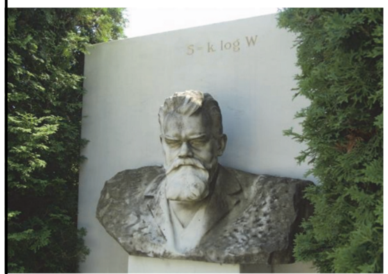
e.g., $Z_{\rm int}^{(i)} = \sum_{j=0}^{\infty} g_{ij}\exp(-E_{ij}/kT)$ of species i ; $Z_{\rm int} = \prod_i Z_{\rm int}^{(i)}$

The free energy $F(T,V,\{N_i\})$ is then obtained from:

$$F(T,V,\{N\}) \;=\; -kT\ln(Z)$$

Statistical mechanics - thermodynamics

$$S = k\ln W \;\;= -k\sum p_i \ln p_i$$



Ludwig Boltzmann
(1844 - 1906)

Probability

$$p_i = \frac{1}{Z}\exp\left(-E_i/kT\right)$$

Partition function (canonical)

$$Z = \sum \exp\left(-E_i/kT\right)$$

Helmholtz free energy F

$$F = -kT\ln Z$$

The equation of state of stellar matter

A self-consistent approximate approach

Partition function: $Z = Z_\mathrm{e}\, Z_\mathrm{trans}\, Z_\mathrm{int}\, Z_\mathrm{rad}\, Z_\mathrm{conf}$

↓

Free energy: $F = F_\mathrm{e} + F_\mathrm{trans} + F_\mathrm{int} + F_\mathrm{rad} + F_\mathrm{conf}$

- F_e: contribution from free electrons (including effects of degeneracy, as appropriate)
- F_trans: contribution from the motion of heavy particles
- F_int: contribution from the internal states in atoms and ions
- F_rad: contribution from radiation
- F_conf: the 'configuration' contribution, resulting from the finite size of atoms and ions, and the Coulomb interaction.

The equation of state of stellar matter

A self-consistent approximate approach

Saha equation can be derived from minimizing free energy *F*(*T*,*V*,{*Ni*}) (e.g. Däppen & Guzik (2000)).

Additional 'corrections', such as the electron chemical potential, $\Delta\mu$, can than easily and consistently be added to *F* by ΔF .

$$\frac{n_{r+1}}{n_r} n_\mathrm{e} = \frac{u_{r+1}}{u_r} 2 \frac{(2\pi m_\mathrm{e} k_\mathrm{B} T)^{3/2}}{h^3} \mathrm{e}^{-\chi_r/k_\mathrm{B}T + \Delta\mu}$$

$$\Delta\mu = -\frac{1}{k_\mathrm{B}T} \left(\frac{\partial \Delta F}{\partial n_\mathrm{e}}\right)_{T,V} .$$

The equation of state of stellar matter

A self-consistent approximate approach

Tackling the problem of the divergent partition function Z_int

$$F_\mathrm{int} = -k_\mathrm{B}T \sum_k \sum_j \ln \left[\sum_i w_{ijk} g_{ijk} \exp(-E_{ijk}/k_\mathrm{B}T) \right] .$$

k …. nr. of elements

j …. nr. of ionization states of each element

i …. nr. of bound (energy) states of each element

w_{ijk} …. newly introduced weights describing probability that state exists (MHD EOS; Mihalas, Hummer, Däppen 1988)

The equation of state of stellar matter

A self-consistent approximate approach

$$F_\mathrm{conf} = F_\mathrm{FV} + F_\mathrm{DH} .$$

F_FV …. Finite volume of atoms and ions → "pressure (density) ionization"

F_DH …. Debye-Hückel approximation for Coloumb effects (screening effect through electrostatic potential of ions)

$$F_\mathrm{DH} = -\frac{k_\mathrm{B}TV}{12\pi r_\mathrm{D}^3} ,$$

$$r_\mathrm{D} \simeq \left[\frac{k_\mathrm{B}T}{4\pi(n_\mathrm{e}e^2 + n_\mathrm{i}e^2)}\right]^{1/2}$$ …. Debye length