

Ionization

- in the previous chapter we assumed complete ionization of all atoms.
- near the surface where $T \uparrow$, the degree of ionization becomes smaller; e.g. Sun: H and He is essentially neutral in the outer atmospheric layers.
- partially ionized gas will have a different μ and essentially all TD quantities will depend on the degree of ionization.

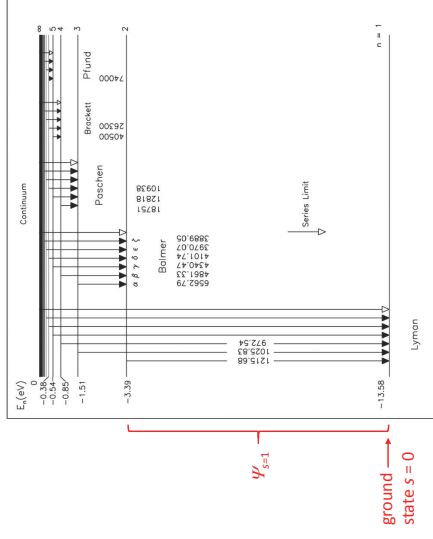
Boltzmann and Saha formulae

- consider partially ionized atoms of a certain element in a unit volume gas in TD equilibrium (TDE). These atoms are distributed over many excitation states s , separated from the ground state $s = 0$ by the energy difference ψ_s .

N. Bohr: Ionization

$$E_n = -\frac{Z^2 m_e c^2 \alpha^2}{2n^2}$$

Hydrogen atom (excitation states s)



Ionization

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Boltzmann and Saha formulae

- consider partially ionized atoms of a certain element in a unit volume gas in TD equilibrium (TDE). These atoms are distributed over many excitation states s , separated from the ground state $s = 0$ by the energy difference ψ_s .
- These excitation states s can be degenerate such that one such state s consists of g_s substates, the so-called statistical weight g_s .
- in equilibrium the number ratio n_s/n_0 between upward and downward transitions through, for example, emission and absorption of photons is

$$\frac{n_s}{n_0} = \frac{g_s}{g_0} e^{-\psi_s/kT} \quad \dots \text{ Boltzmann formula}$$

Statistical mechanics - thermodynamics

$$S = k \ln W = -k \sum p_i \ln p_i$$



Ludwig Boltzmann (1844 - 1906)

Probability

$$p_i = \frac{1}{Z} \exp(-E_i/kT)$$

Partition function (canonical)

$$Z = \sum \exp(-E_i/kT)$$

Helmholtz free energy F

$$F = -kT \ln Z$$

Ionization

$$\frac{n_s}{n_0} = \frac{g_s}{g_0} e^{-\psi_s/kT}, \quad \dots \text{ Boltzmann formula}$$

which describes the distribution of particles over states of different energy.

With the **total number of atoms** $n = \sum_s n_s$ the Boltzmann formula becomes, after summation of all energy (excitation) s states

$$g_0 \frac{n}{n_0} = g_0 \sum_{s=0}^{\infty} \frac{n_s}{n_0} = g_0 + g_1 e^{-\psi_1/kT} + g_2 e^{-\psi_2/kT} + \dots := u_p,$$

where $u_p = u_p(T)$ is the so-called partition function.

Boltzmann's formula can then be rewritten as

$$\frac{n_s}{n} = \frac{g_s}{u_p} e^{-\psi_s/kT}.$$

We have to distinguish between **excited states s** (bound electrons distributed over discrete states) and **ionization** (free electrons with a continuous manifold of states).

Ionization

$$\frac{n_s}{n} = \frac{g_s}{u_p} e^{-\psi_s/kT}.$$

- We have to distinguish between **excited states s** (bound electrons distributed over discrete states) and **ionization** (free electrons with a continuous manifold of states).

- ions (atoms) that already lost r electrons are said to be in the r^{th} ionization state.
- the **energy necessary** to remove the next electron (e^-) from the **ground state** is then χ_r .
- the free e^- will have a kinetic energy $p_e^2/(2m_e)$ relative to the atom, so its total energy relative to its bound state is therefore $\chi_r + p_e^2/(2m_e)$, where p_e is the (free) electron's momentum relative to the ion, which is now in the ionization state $r+1$.
- consider the **lower state** as an r -times ionized ion in the **ground state**, and the **upper state** be that of the $(r+1)$ -times ion plus free e^- with p_e in $[p_e, p_e+dp_e]$.
- with number densities n_r and dn_{r+1} of the two states and with the statistical weight for the upper state being the product of g_{r+1} of the ion and $dg(p_e)$ of the free e^- , we have in TDE

$$\frac{dn_{r+1}}{n_r} = \frac{g_{r+1} dg(p_e)}{g_r} \exp\left(-\frac{\chi_r + p_e^2/(2m_e)}{kT}\right).$$

Ionization

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- **What is the statistical weight $dg(p_e)$** in the momentum interval $[p_e, p_e+dp_e]$?

Pauli principle: (6-dim.) phase space of volume $dq_1 dq_2 dq_3 dp_1 dp_2 dp_3 = dV d^3p$ can only contain up to $2dV d^3p/h^3$ electrons, i.e. up to 2 e^- /quantum cell of volume h^3 .

$$\rightarrow dg(p_e) = \frac{2dV d^3p_e}{h^3}.$$

- the **volume available per e^-** is $dV = 1/n_e$, while the volume available in the **momentum interval $[p_e, p_e+dp_e]$** is $d^3p_e = 4\pi p_e^2 dp_e$, since all these points are on a spherical shell with radius p_e and thickness dp_e .

$$\rightarrow dg(p_e) = \frac{8\pi p_e^2 dp_e}{n_e h^3} \quad \text{and} \quad \frac{dn_{r+1}}{n_r} = \frac{g_{r+1}}{g_r} \frac{8\pi p_e^2 dp_e}{n_e h^3} \exp\left(-\frac{\chi_r + p_e^2/(2m_e)}{kT}\right).$$

Ionization

$$dg(p_e) = \frac{8\pi p_e^2 dp_e}{n_e h^3} \quad \text{and} \quad \frac{dn_{r+1}}{n_r} = \frac{g_{r+1}}{g_r} \frac{8\pi p_e^2 dp_e}{n_e h^3} \exp\left(-\frac{\chi_r + p_e^2/(2m_e)}{kT}\right).$$

All upper states (ions of ionization degree $r+1$ in the **ground state** and free e^- of all momenta) are then obtained from integrating over p_e :

$$\frac{n_{r+1}}{n_r} = \frac{g_{r+1}}{g_r} \frac{8\pi}{n_e h^3} e^{-\chi_r/kT} \int_0^{\infty} p_e^2 \exp\left(-\frac{p_e^2}{2m_e kT}\right) dp_e.$$

With $\int_0^{\infty} x^2 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{4\alpha^{3/2}}$ for $\alpha > 0$ we obtain

$$\frac{n_{r+1}}{n_r} n_e = \frac{g_{r+1}}{g_r} f_r(T), \quad \text{where} \quad f_r(T) = 2 \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_r/kT}.$$

Saha equation (Meghnad Saha, 1920) [only for the ground state].

Ionization

state of ionization

$$n_{i,k}$$

state of excitation

The **Saha equation** (for the **ground state only**) is than written more precisely as

$$\frac{n_{r+1,0}}{n_{r,0}} n_e = \frac{g_{r+1,0}}{g_{r,0}} f_r(T), \text{ where } f_r(T) = 2 \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_r/kT}.$$

The number density of ions in the ionization state r (and in all excitation states s) is

$$n_{r,\cdot} = \sum_s n_{r,s},$$

and, as before with the definition for all ions n , the **Boltzmann formula for ions of state r** becomes

$$\frac{n_{r,s}}{n_{r,0}} = \frac{g_{r,s}}{g_{r,0}} e^{-\psi_{r,s}/kT},$$

where $\psi_{r,s}$ is the **excitation energy** of (one single) state s of ions in ionization state r .

Ionization

$$\frac{n_{r+1}}{n_r} n_e = \frac{g_{r+1}}{g_r} f_r(T), \text{ where } f_r(T) = 2 \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_r/kT}.$$

Saha equation (Meghnad Saha, 1920) [only for the ground state].

In order to be more precise, i.e. **indicating only the ground state $s=0$** , we use the quantities

$$n_{r+1,0}, n_{r,0}, g_{r+1,0}, g_{r,0},$$

where the second subscript indicates the ground state for which these quantities are defined.

By the notation $n_{r+1}, n_r, g_{r+1}, g_r$ we mean **from now on** the number densities of ions and statistical weights of **all states of excitation**, where as

↖ **all states of excitation @ $r+1$ ionization**

A particular state of excitation is indicated by a second subscript such that

state of ionization

$$n_{i,k}$$

state of excitation

is the number density of atoms in the **stage i of ionization** and in **state k of excitation**, and $g_{i,k}$ is the corresponding statistical weight.

Ionization

number of ionized H atoms

$$x = \frac{n_1}{n_0 + n_1}, \quad [n_1/n_0 = x/(1-x) \cdot 1]$$

number of neutral H atoms

If the gas is neutral, then $x = 0$;
if it is completely ionized, $x = 1$.

With this definition, the l.h.s. of Saha equation can be replaced by

$$\frac{n_{r+1}}{n_r} P_e = \frac{x P_e}{1-x},$$

and if $n = n_0 + n_1$ is the total number of H atoms, we obtain for the partial pressure P_e of the e- to the **total gas pressure** P_{gas} :

$$P_e = n_e kT = (n + n_e) kT \frac{n_e}{n + n_e} = P_{\text{gas}} \frac{n_e}{n + n_e}.$$

Ionization

the **Boltzmann formula for ions of state r** becomes

$$\frac{n_{r,s}}{n_{r,0}} = \frac{g_{r,s}}{g_{r,0}} e^{-\psi_{r,s}/kT}, \text{ where } \psi_{r,s} \text{ is the excitation energy of (one single) state } s.$$

The number density of ions of ionization state r , $n_r = \sum_s n_{r,s}$, can then be written as

$$\frac{g_{r,0}}{n_{r,0}} n_r = g_{r,0} \sum_s \frac{n_{r,s}}{n_{r,0}} = g_{r,0} + g_{r,1} e^{-\psi_{r,1}/kT} + g_{r,2} e^{-\psi_{r,2}/kT} + \dots := u_r,$$

$\rightarrow n_r g_{r,0} = n_{r,0} u_r$

where $u_r = u_r(T)$ is the partition function for the ion in state r .

With $n_r g_{r,0} = n_{r,0} u_r$, the **Saha equation eventually becomes for all stages of excitation**

$$\frac{n_{r+1}}{n_r} n_e = \frac{u_{r+1}}{u_r} f_r(T), \quad f_r(T) = 2 \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_r/kT}.$$

And finally with $P_e = n_e kT$

$$\frac{n_{r+1}}{n_r} P_e = \frac{u_{r+1}}{u_r} 2 \frac{(2\pi m_e)^{3/2}}{h^3} (kT)^{5/2} e^{-\chi_r/kT}.$$

Ionization

total number of H atoms

$$P_e = n_e kT = \frac{n_e}{n + n_e} kT = \frac{P_{\text{gas}}}{n + n_e} \frac{n_e}{n}$$

For each ionized atom there is just one electron ($n_e = n_1$); therefore $x := \frac{n_e}{n}$ & number of ionized H atoms

$$P_e = \frac{x}{1+x} P_{\text{gas}}$$

and the **Saha equation** can be written in the form with $n_{r+1}/n_r = n_1/n_0 = x/(1-x)$

$$\frac{x^2}{1-x^2} = K_H, \quad \text{with } K_H = \frac{u_1}{u_0} \frac{2}{P_{\text{gas}}} \frac{(2\pi m_e)^{3/2}}{h^3} (kT)^{5/2} e^{-\chi_H/kT},$$

where $\chi_H = 13.6 \text{ eV}$ is the ionization energy of hydrogen;

quadratic equation for x , which can be solved for given T and P_{gas} .

We still need, however, the **partition function u** .

Ionization

$$\frac{x^2}{1-x^2} = K_H, \quad \text{with } K_H = \frac{u_1}{u_0} \frac{2}{P_{\text{gas}}} \frac{(2\pi m_e)^{3/2}}{h^3} (kT)^{5/2} e^{-\chi_H/kT}.$$

Partition function u :
statistical weights from quantum mechanics (e.g. Cox, 2000, pp. 2-34):

ground state: $u_0 \approx g_{0,0} = 2$,
ionized H : $u_1 = 1$

Example: solar atmosphere : $P_{\text{gas}} = 1.01 \times 10^5 \text{ dyn cm}^{-2}$ } $x = 5 \times 10^{-5}$,
 $T = 5779 \text{ K}$

deeper layers : $P_{\text{gas}} = 3.35 \times 10^{12} \text{ dyn cm}^{-2}$ } $x = 0.985$,
 $T = 7.17 \times 10^5 \text{ K}$ } H is nearly completely ionized.

Because K_H increases with T (decreases with P_{gas}) and l.h.s. increases with x : x increases with T and decreases with P_{gas} .

Ionization

In order to determine mean mol weight μ for the H gas with ionization degree x , we define E as the number of free electrons per atom (neutral or ionized), which is here (for H) simply

$$E = \frac{n_e}{n} = x = \frac{\mu_0}{\mu_e} \rightarrow \frac{1}{1+E} = \frac{n}{n+n_e}.$$

The density $\rho = (n + n_e) \mu m_0 = n(\mu_0 m_0 + n_e \mu_e m_0) \rightarrow \mu = \frac{\rho}{m_0(n+n_e)}$.

average particle mass per free particle, per nucleus, pre free electron

With the definition of E and $n = n_0 + n_1$ we obtain for the mean molecular weight

$$\mu = \frac{\rho}{m_0 n} \frac{1}{1+E} = \frac{\mu_0}{1+E} = \mu_e \frac{E}{1+E},$$

for a neutral gas ($E=0$)

where we have neither replaced μ_0 by its value 1 for hydrogen nor E by x , since above expression(s) also holds for a mixture of gases.

Ionization

TD quantities for a pure hydrogen gas

$$\delta = -(\partial \ln \rho / \partial \ln T)_P :$$

for $x = 0 \rightarrow \delta = 1$ } because $\mu = \text{constant}$ in both cases.
 $x = 1 \rightarrow \delta = 1$ }

for any $x \rightarrow \delta = 1 + \frac{1}{1+E} \left(\frac{\partial E}{\partial \ln T} \right)_P$, ($\mu_0 = \text{constant}$)

which also holds for a mixture of gases.

For pure hydrogen $E = x$

$$\delta = 1 + \frac{1}{2} x(1-x) \left(\frac{5}{2} + \frac{\chi_H}{kT} \right), \quad [F(\beta) \propto (1-\beta)/\beta].$$

Note: for this last expression we needed the derivative of x : obtained from diff. Saha equ.

Ionization

TD quantities for a pure hydrogen gas

$$\nabla_{\text{ad}} = \frac{P\delta}{T\rho c_P} = \frac{2+x(1-x)\phi_{\text{H}}^2}{5+x(1-x)\phi_{\text{H}}^2}$$

Ionization

TD quantities for a pure hydrogen gas

$$c_P := \left(\frac{dq}{dT}\right)_P = \left(\frac{\partial u}{\partial T}\right)_P + P \left(\frac{\partial v}{\partial T}\right)_P$$

Using $P = \Re\rho T/\mu$ provides $c_P = \left(\frac{\partial u}{\partial T}\right)_P + \frac{\Re}{\mu}\delta$,

for which we need the internal energy / unit mass $u = \frac{3}{2}\frac{\Re}{\mu_0}(1+E)T + u_{\text{ion}}$,
kinetic energy of ions & e- ionization/recombination energy

For pure hydrogen, $E = x$ and $u_{\text{ion}} = x\chi_{\text{H}}/(\mu_0 m_{\text{H}}) = x\chi_{\text{H}}/m_{\text{H}}$,
 $\mu_0 = 1$ for H

$$\frac{\mu_0}{\Re} = \frac{5}{2}(1+x) + \frac{\phi_{\text{H}}^2}{G(x)} \quad \text{with} \quad \phi_{\text{H}} := \frac{5}{2} + \frac{\chi_{\text{H}}}{kT}$$

and $G(x) := \frac{1}{x(1-x)} + \frac{1}{x(1-x^2)} = \frac{2}{x(1-x^2)}$.

Ionization

Hydrogen-helium mixtures

$$x_{\text{H}}^0 = \frac{n_{\text{H}}^0}{n_{\text{H}}}, \quad x_{\text{H}}^1 = \frac{n_{\text{H}}^1}{n_{\text{H}}}, \quad x_{\text{He}}^0 = \frac{n_{\text{He}}^0}{n_{\text{He}}}, \quad x_{\text{He}}^1 = \frac{n_{\text{He}}^1}{n_{\text{He}}},$$

$$x_{\text{He}}^2 = \frac{n_{\text{He}}^2}{n_{\text{He}}}, \quad x_{\text{He}}^{\text{ion}} = \frac{n_{\text{He}}^{\text{ion}}}{n_{\text{He}}},$$

with $n_{\text{H}} = n_{\text{H}}^0 + n_{\text{H}}^1$
 $n_{\text{He}} = n_{\text{He}}^0 + n_{\text{He}}^1 + n_{\text{He}}^2$

and where the n_i are number densities of ions of type i in ionization state r .

Contribution to internal energy / unit mass:

$$u_{\text{ion}} = \frac{1}{m_{\text{H}}} \left\{ X x_{\text{H}}^1 \chi_{\text{H}} + \frac{1}{4} Y \left[x_{\text{He}}^1 \chi_{\text{He}} + x_{\text{He}}^2 (\chi_{\text{He}} + \chi_{\text{He}}^{\text{ion}}) \right] \right\},$$

since $X/m_{\text{H}}, Y/(4m_{\text{H}})$ are the numbers of hydrogen and helium atoms (neutral and ionized) per unit mass.

Ionization

Hydrogen-helium mixtures

- weight fractions X, Y
- six types of particles: neutral and ionized hydrogen; neutral, ionized, and double ionized helium; and electrons.
- three types of ionization energy: χ_{H}^0 for hydrogen ($\chi_{\text{H}}^0 = 13.598 \text{ eV}$),
 $\chi_{\text{He}}^0, \chi_{\text{He}}^1$ for neutral and single ionized helium
 $(\chi_{\text{He}}^0 = 24.587 \text{ eV}, \chi_{\text{He}}^1 = 54.418 \text{ eV})$.
contribute to u_{ion}

By $x_{\text{H}}^0, x_{\text{H}}^1, x_{\text{He}}^0, x_{\text{He}}^1, x_{\text{He}}^2$ we define **degrees of ionization**,
i.e. x_i^r gives the number of atoms of type i in ionization state r ($= r$ electrons lost) divided by the total number of atoms (irrespective of their state of ionization):

$$x_{\text{H}}^0 = \frac{n_{\text{H}}^0}{n_{\text{H}}}, \quad x_{\text{H}}^1 = \frac{n_{\text{H}}^1}{n_{\text{H}}}, \quad x_{\text{He}}^0 = \frac{n_{\text{He}}^0}{n_{\text{He}}},$$

$$x_{\text{He}}^1 = \frac{n_{\text{He}}^1}{n_{\text{He}}}, \quad x_{\text{He}}^2 = \frac{n_{\text{He}}^2}{n_{\text{He}}},$$

Ionization

Hydrogen-helium mixtures

- number E of free electrons per atom

$$E = \left[Xx_H^1 + \frac{1}{4}Y(x_{He}^1 + 2x_{He}^2) \right] \mu_0$$

$$\frac{x_H^1}{x_H^0} \frac{E}{E+1} = K_H^0, \quad \frac{x_{He}^1}{x_{He}^0} \frac{E}{E+1} = K_{He}^0, \quad \frac{x_{He}^2}{x_{He}^1} \frac{E}{E+1} = K_{He}^1,$$

$$x_H^0 + x_H^1 = 1, \quad x_{He}^0 + x_{He}^1 + x_{He}^2 = 1.$$

- with X, Y, P_{gas} , and T to be given, the **six equations** above provide solutions to the **six unknowns** $x_H^0, x_H^1, x_{He}^0, x_{He}^1, x_{He}^2, E$.

- the coupling of the (Saha) equations through E makes an **analytical treatment impossible**: the degrees of ionization **have to be obtained numerically**, typically by iteration, starting with a trial value for E .

Ionization

Hydrogen-helium mixtures

- number E of free electrons per atom

$$E = \left[Xx_H^1 + \frac{1}{4}Y(x_{He}^1 + 2x_{He}^2) \right] \mu_0 = \frac{\mu_0}{\mu_e} = \sum_i \frac{\mu_0 X_i}{\mu_i} \sum_{r=0}^{Z_i} x_i^r$$

- we have now **three Saha equations**:

$$\frac{x_H^1}{x_H^0} \frac{E}{E+1} = K_H^0, \quad \frac{x_{He}^1}{x_{He}^0} \frac{E}{E+1} = K_{He}^0, \quad \frac{x_{He}^2}{x_{He}^1} \frac{E}{E+1} = K_{He}^1,$$

with

$$K_i^r = \frac{u_i^{r+1}}{u_i^r} \frac{2}{P_{\text{gas}}} \frac{(2\pi m_e)^{3/2} (kT)^{5/2}}{h^3} e^{-x_i/kT} \quad \text{for } i = \text{H, He},$$

and

$$x_H^0 + x_H^1 = 1, \quad x_{He}^0 + x_{He}^1 + x_{He}^2 = 1.$$

Ionization

General case

number of free e- per atom $\rightarrow E = \sum_i \nu_i x_i^r = \sum_i \frac{Z_i}{\mu_i} \sum_{r=0}^{Z_i} x_i^r$

charge number $\rightarrow \sum_i \nu_i x_i^r = \sum_i \frac{Z_i}{\mu_i} \sum_{r=0}^{Z_i} x_i^r$

ionization degree (number of atoms, in ion. state r over total number of atoms,) $\rightarrow \sum_i \frac{Z_i}{\mu_i} \sum_{r=0}^{Z_i} x_i^r$

atoms of type $i \rightarrow \sum_i \nu_i x_i^r$

molecular weight $\rightarrow \sum_i \frac{\mu_0 X_i}{\mu_i} \sum_{r=0}^{Z_i} x_i^r$

(rel.) mass fraction $\rightarrow \sum_i \frac{Z_i}{\mu_i} \sum_{r=0}^{Z_i} x_i^r$

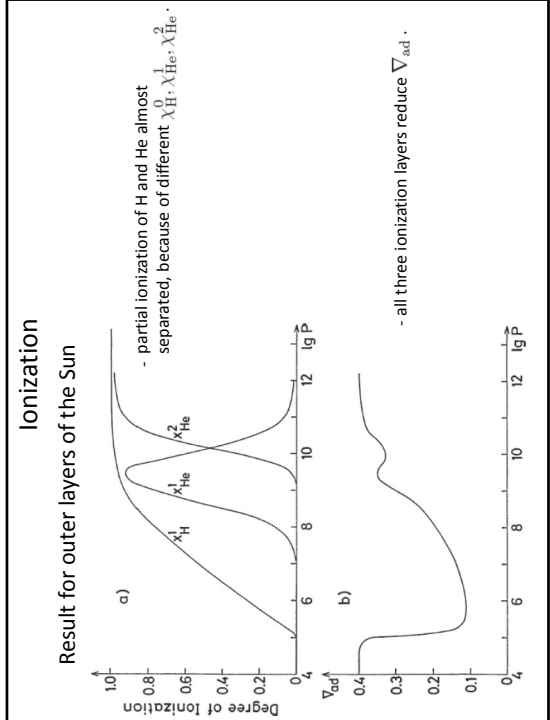
r electrons lost

where $\nu_i = n_i/n = X_i \mu_0 / \mu_i$ is the relative number of particles of type i .

- degrees of ionization are obtained from the set of Saha equations

$$\frac{x_i^{r+1}}{x_i^r} \frac{E}{E+1} = K_i^r, \quad i = 1, 2, \dots, \quad r = 0, 1, \dots, Z_i,$$

where the K_i^r are given by $K_i^r = \frac{u_i^{r+1}}{u_i^r} \frac{2}{P_{\text{gas}}} \frac{(2\pi m_e)^{3/2} (kT)^{5/2}}{h^3} e^{-x_i/kT}$.



General case

Ionization

number of free e- per atom $E = \sum_i \nu_i x_i^r = \sum_i \nu_i \sum_{r=0}^{Z_i} X_i \sum_{r=0}^{Z_i} x_i^r r$

charge number Z_i

ionization degree (numbers of atoms, in ion. state r over total number of atoms,)

molecular weight μ_i

(rel.) mass fraction

atoms of type i

r electrons lost

where $\nu_i = n_i/n = X_i \mu_0 / \mu_i$ is the relative number of particles of type i .

In addition we have the relations $\sum_{r=0}^{Z_i} x_i^r = 1, \quad i = 1, 2, \dots,$

which for a given type i of atoms, together with the Z_i Saha equations, in which E is replaced by the expression given above, represent $Z_i + 1$ equations for the $Z_i + 1$ degrees r of ionization, i.e. there are as many equations as variables; the equations can be solved iteratively to obtain the degrees of ionization and eventually the mean molecular weight $\mu = \mu_0 / (1 + E)$.

General case

Ionization

number of free e- per atom $E = \sum_i \nu_i x_i^r = \sum_i \nu_i \sum_{r=0}^{Z_i} X_i \sum_{r=0}^{Z_i} x_i^r r$

charge number Z_i

ionization degree (numbers of atoms, in ion. state r over total number of atoms,)

molecular weight μ_i

(rel.) mass fraction

atoms of type i

r electrons lost

The kinetic part of the internal energy u is

$$u_{\text{kin}} = \frac{3}{2} \mathcal{R} T = \frac{3}{2} \frac{\mathcal{R}}{\mu_0} (1 + E) T,$$

while the ionization energy per mass unit, is

$$u_{\text{ion}} = \sum_i \frac{X_i}{\mu_i m_H} \sum_{r=0}^{Z_i} x_i^r \sum_{s=0}^{r-1} \chi_i^s,$$

which is the general form of

$$u_{\text{ion}} = \frac{1}{m_H} \left\{ X_{\text{H}} \chi_{\text{H}}^0 + \frac{1}{4} Y \left[x_{\text{He}}^1 \chi_{\text{He}}^0 + x_{\text{He}}^2 (\chi_{\text{He}}^0 + \chi_{\text{He}}^1) \right] \right\},$$

Ionization

Limitation of the Saha formula

Let's consider pure H only.

Example: solar atmosphere : $P_{\text{gas}} = 1.01 \times 10^5 \text{ dyn cm}^2$ } $x = 5 \times 10^{-5}$,
 $T = 5779 \text{ K}$

deeper layers : $P_{\text{gas}} = 3.35 \times 10^{12} \text{ dyn cm}^2$ } $x = 0.985$,
 $T = 7.17 \times 10^5 \text{ K}$ } H is nearly completely ionized.

solar center : $P_c \simeq P_{\text{gas}} \simeq 2.32 \times 10^{17} \text{ dyn cm}^{-2}$ } $x \simeq 0.8$,
 $T_c \simeq 1.57 \times 10^7 \text{ K}$ } 20% of H would be neutral (this must be wrong!)

Ionization

Limitation of the Saha formula

solar center : $P_c \simeq P_{\text{gas}} \simeq 2.32 \times 10^{17} \text{ dyn cm}^{-2}$ } $x \simeq 0.8$,
 $T_c \simeq 1.57 \times 10^7 \text{ K}$ } 20% of H would be neutral (this must be wrong!)

$$\frac{x^2}{1-x^2} = K_{\text{H}}, \quad \text{with} \quad K_{\text{H}} = \frac{u_1}{u_0} \frac{2 (2\pi m_e)^{3/2}}{P_{\text{gas}}} (kT)^{5/2} e^{-\chi_{\text{H}}/kT}.$$

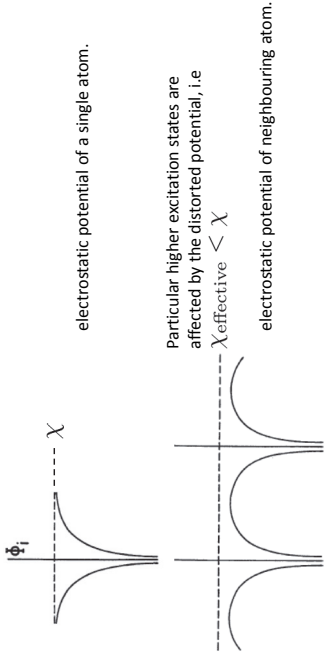
For $T \gg \chi$, exponential ~ 1 and for $\nabla \approx d \ln T / d \ln P_{\text{gas}} < 2/5$, x decreases inwards because of dominating increase of P_{gas} .

This is a result of the Saha equation of **not knowing the size of the atoms!**

Ionization

Limitation of the Saha formula

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Ionization

Limitation of the Saha formula

This is a result of the Saha equation of not knowing the **size of the atoms!**

Lets' estimate the **mean distance d** between two atoms:

$$\frac{4d^3\pi}{3} \approx \frac{1}{n_H} \rightarrow \text{number density of H atoms (cm}^{-3}\text{)}$$

We also know from quantum mechanics the orbital radius a of the (bound) electron:

$$a = a_0\nu^2,$$

where $a_0 = 5.3 \times 10^{-9}$ cm is the Bohr radius, and ν the quantum number ($\nu = n$).

Sun: $\rho_c \approx 150 \text{ g/cm}^3$,
 $n_H \approx \rho_c/m_H \approx 10^{26} \text{ cm}^{-3}$

There will only be bound states if $a < d/2$

or
$$\nu^2 < \left(\frac{3}{4\pi n_H}\right)^{1/3} \frac{1}{2a_0}.$$

\rightarrow so-called pressure ionization.

Saha equation valid for $d > 10a_0 \rightarrow \rho = \mu_0 n_H n_{\text{ion}} < \frac{3\rho_0 m_H}{4\pi(10a_0)^3} = 2.66 \times 10^{-3} \mu_0 \text{ g cm}^{-3}.$