

Spin- $\frac{1}{2}$ field

Bispinors

The lowest irreducible representation of the full Lorentz group $O(1,3)$, containing spin- $\frac{1}{2}$, is

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right). \quad (1)$$

The matrices from this representation transform four-component objects called *bispinors* or Dirac spinors,

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \quad (2)$$

where the two-component objects $\phi \in (\frac{1}{2}, 0)$ and $\chi \in (0, \frac{1}{2})$ are the ("right" and "left") *Weyl spinors*. Their infinitesimal transformation matrices are

$$\mathbf{t}_{(\frac{1}{2}, 0)} = 1 + i\frac{1}{2}\vec{\sigma}d\vec{w}, \quad (3)$$

and

$$\mathbf{t}_{(0, \frac{1}{2})} = 1 + i\frac{1}{2}\vec{\sigma}d\vec{w}^*. \quad (4)$$

Apparently,

$$\mathbf{t}_{(0, \frac{1}{2})} = \left(\mathbf{t}_{(\frac{1}{2}, 0)}^\dagger\right)^{-1}, \quad (5)$$

and therefore under a Lorentz transformation Λ

$$\phi \xrightarrow{\Lambda} \mathbf{t}\phi, \quad \chi \xrightarrow{\Lambda} \mathbf{t}^\dagger{}^{-1}\chi, \quad (6)$$

where $\mathbf{t} \equiv \mathbf{t}_{(\frac{1}{2}, 0)}$.

Under parity transformation, \mathcal{P} , the Weyl spinors transform into each other,

$$\phi \xleftrightarrow{\mathcal{P}} \chi. \quad (7)$$

Bilinear forms of bispinors

The Lagrangian for the field must be a real scalar, bilinear in the field. That is, it has to be built out of $\psi^+ \otimes \psi$ which transforms under the representation

$$\left(\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right)^* \otimes \left(\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)\right). \quad (8)$$

According to the general rule this direct product reduces to a direct sum of the following representations,

$$(0, 0); (0, 1) \oplus (1, 0); \left(\frac{1}{2}, \frac{1}{2}\right), \quad (9)$$

where the first is scalar, the second – antisymmetric tensor, and the third – four-vector.

Scalars

Using the transformation laws (6) and (7) it is easy to show that the (real) scalar is the combination

$$\bar{\psi}\psi \equiv \phi^+\chi + \chi^+\phi, \quad (10)$$

where $\bar{\psi} \equiv \psi^\dagger\gamma^0$ and the block-matrix γ^0 is given as

$$\gamma^0 = \begin{bmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{bmatrix}. \quad (11)$$

The combination

$$i\bar{\psi}\gamma^5\psi \equiv -i(\phi^+\chi - \chi^+\phi), \quad (12)$$

where

$$\gamma^5 = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{bmatrix} \quad (13)$$

is a (real) *pseudo-scalar*, that is, it changes sign under parity transformation.

Vectors

Since $\phi \in (\frac{1}{2}, 0)$ and $\phi^* \in (0, \frac{1}{2})$, their direct product is a four-vector,

$$\phi^* \otimes \phi = \left(\frac{1}{2}, \frac{1}{2}\right). \quad (14)$$

Apparently, this four-vector is

$$\{\phi^+\phi, \phi^+\vec{\sigma}\phi\}. \quad (15)$$

Analogously,

$$\{\chi^+\chi, \chi^+\vec{\sigma}\chi\} \quad (16)$$

is also a four-vector.

Now we can build a *polar vector*,

$$\bar{\psi}\gamma^\alpha\psi \equiv \{\phi^+\phi + \chi^+\chi, \phi^+\vec{\sigma}\phi - \chi^+\vec{\sigma}\chi\}, \quad (17)$$

and an *axial- or pseudo-vector*,

$$i\bar{\psi}\gamma^a\gamma^5\psi \equiv i\{\phi^+\phi - \chi^+\chi, \phi^+\vec{\sigma}\phi + \chi^+\vec{\sigma}\chi\}, \quad (18)$$

where $\gamma^a = \{\gamma^0, \vec{\gamma}\}$ and

$$\vec{\gamma} = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix}. \quad (19)$$

Antisymmetric tensor

The representation $(0, 1)$ can be made of

$$\phi^* \otimes \chi \sim \left(0, \frac{1}{2}\right) \otimes \left(0, \frac{1}{2}\right) \ni (0, 1), \quad (20)$$

and, correspondingly, $(1, 0)$ out of

$$\chi^* \otimes \phi \sim \left(\frac{1}{2}, 0\right) \otimes \left(\frac{1}{2}, 0\right) \ni (1, 0). \quad (21)$$

Under rotations the objects $(0, 1)$ and $(1, 0)$ transform as pure vectors. Apparently they are

$$\phi^+\vec{\sigma}\chi \sim (0, 1) \quad (22)$$

and

$$\chi^+ \vec{\sigma} \phi \sim (1, 0). \quad (23)$$

Under reflections these two objects transform into each other. Therefore their sum and difference is a six-component objects that transforms through itself under the full Lorentz group $O(1, 3)$.

Using *gamma*-matrices this tensor is usually written as

$$\frac{1}{2} \bar{\psi} (\gamma^a \gamma^b - \gamma^b \gamma^a) \psi. \quad (24)$$

Resume

The direct product $\psi^* \otimes \psi$ of two bispinors can be reduced into five irreducible covariant objects:

1. scalar, $\bar{\psi} \psi$;
2. pseudo-scalar, $i \bar{\psi} \gamma_5 \psi$;
3. vector, $\bar{\psi} \gamma^a \psi$;
4. pseudo-vector, $i \bar{\psi} \gamma^a \gamma_5 \psi$;
5. antisymmetric tensor, $\frac{1}{2} \bar{\psi} (\gamma^a \gamma^b - \gamma^b \gamma^a) \psi$,

where $\bar{\psi} \equiv \psi^\dagger \gamma_0$ and the block-matrices γ^a and γ^5 are called *gamma*-matrices or Dirac matrices.

Gamma matrices in Dirac basis

The bispinors and gamma-matrices in the previous section are given in the so-called Weyl or *chiral* basis. A similarity transformation

$$\begin{aligned} \psi &\rightarrow S \psi \\ \gamma^a &\rightarrow S \gamma^a S^{-1} \end{aligned} \quad (25)$$

with the matrix

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (26)$$

changes Weyl basis into Dirac or *standard* basis, where the γ -matrices are

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (27)$$

$$\vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix} \quad (28)$$

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (29)$$

The γ -matrices satisfy the anti-commutation relation

$$\gamma^a \gamma^b + \gamma^b \gamma^a = 2g^{ab}, \quad (30)$$

where g^{ab} is the Minkowski metric tensor. The matrix γ_5 anti-commutes with γ^a ,

$$\{\gamma^a, \gamma_5\} = 0. \quad (31)$$

Lagrangian of spin- $\frac{1}{2}$ field

The Lagrangian must be a real scalar, bilinear in ψ and $\partial_a \psi$. A suitable form is

$$\mathcal{L} = \frac{i}{2} (\bar{\psi} \gamma^a \partial_a \psi - \partial_a \bar{\psi} \gamma^a \psi) - m \bar{\psi} \psi. \quad (32)$$

Dirac equation

The general expression for the Euler-Lagrange equation,

$$\partial_a \frac{\partial \mathcal{L}}{\partial (\partial_a \bar{\psi})} = \frac{\partial \mathcal{L}}{\partial \bar{\psi}}, \quad (33)$$

with the Lagrangian (32) leads to the so-called *Dirac equation*,

$$(i \gamma^a \partial_a - m) \psi = 0. \quad (34)$$

Energy-momentum tensor and conserved current

Again, using the general expressions, the energy-momentum tensor and the conserved current are directly obtained from the Lagrangian,

$$T_0^0 = -i \bar{\psi} \vec{\gamma} \vec{\partial} \psi + m \bar{\psi} \psi, \quad (35)$$

$$j^a = \bar{\psi} \gamma^a \psi. \quad (36)$$