Reaction amplitude

An archetypal process in quantum field theory is a reaction where two particles are prepared (e.g. in an accelerator) in an initial plane-wave state with given momenta, for example, a boson with momentum $\mathbf{k}$ and a fermion with momentum $\mathbf{p}$ and polarization λ,

$$|i\rangle = a^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{p}\lambda}|0\rangle \;, \tag{1}$$

where $|0\rangle$ is the vacuum state. The particles are then sent toward each other, a reaction happens with certain probability, and the reaction products are detected by the detection system in their final plane-wave states with certain momenta. Suppose the detectors measure the same particles (elastic scattering) with some (presumably changed in the reaction) momenta and polarization,

$$\langle f| = \langle 0|a_{\mathbf{k}'} a_{\mathbf{p}'\lambda'} \;. \tag{2}$$

The reaction amplitude is given by the matrix element

$$S_{fi} = \langle f|S|i\rangle \tag{3}$$

where the S-matrix is written as a perturbation series of T-products of the interaction Lagrangians

$$S = \sum_n \frac{i^n}{n!} \int \prod_{j=1}^{n} d^4x_j T \prod_{j=1}^{n} \mathcal{L}_v(x_j) \;. \tag{4}$$

The reaction probability is given by $|S_{fi}|^2$.

Normal product

The matrix element (3) could be easily calculated if only instead of the T-product there were an arrangement where all generators stand to the left of all annihilators. Indeed since $a|0\rangle = 0$ and $\langle 0|a^\dagger = 0$ the matrix element

$$\langle 0|a_{\mathbf{k}'} a_{\mathbf{p}'\lambda'}(\text{generators})(\text{annihilators})a^\dagger_{\mathbf{k}} a^\dagger_{\mathbf{p}\lambda}|0\rangle \tag{5}$$

is zero unless

$$(\text{generators}) \equiv a^\dagger_{\mathbf{k}'} a^\dagger_{\mathbf{p}'\lambda'} \;, \tag{6}$$

$$(\text{annihilators}) \equiv a_{\mathbf{k}} a_{\mathbf{p}\lambda} \tag{7}$$

in which case it is equal one.

Such ordering is called *normal* or N-product of fields,

$$N(\text{fields}) = (\text{generators})(\text{annihilators}) \;. \tag{8}$$

Propagator

The difference $\Delta(AB)$ (also denoted as $\underline{AB}$) between T- and N-product of two fields A and B is called a *propagator*

$$\Delta(AB) = T(AB) - N(AB) \;. \tag{9}$$

Apparently, if the fields commute, e.g. for a complex scalar field $[\phi, \phi] = [\phi^\dagger, \phi^\dagger] = 0$, there is no difference between T- and N-product.

$$\Delta(\phi\phi) = \Delta(\phi^\dagger\phi^\dagger) = 0 \tag{10}$$

Let us calculate the propagator $\Delta(\phi(x)\phi^\dagger(x'))$ for a complex scalar field ϕ. Introducing the positive- and negative-frequency plane-waves,

$$\phi(x) = \phi^{(+)}(x) + \phi^{(-)}(x) \;, \tag{11}$$

where

$$\phi^{(+)}(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} a_{\mathbf{k}} e^{-ikx} \;, \tag{12}$$

$$\phi^{(-)}(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} b^\dagger_{\mathbf{k}} e^{+ikx} \;, \tag{13}$$

the N-product is given as[1]

$$\begin{aligned} N\Big(\phi(x)\phi^\dagger(x')\Big) &= \phi^{(+)\dagger}(x')\phi^{(+)}(x) \\ &+\phi^{(+)}(x)\phi^{(-)\dagger}(x') + \phi^{(-)}(x)\phi^{(+)\dagger}(x') \\ &+\phi^{(-)}(x)\phi^{(-)\dagger}(x') \;. \end{aligned} \tag{14}$$

The T-product is defined as

$$T\Big(\phi(x)\phi^\dagger(x')\Big) = \begin{cases} \phi(x)\phi^\dagger(x') \;, & t > t' \\ \phi^\dagger(x')\phi(x) \;, & t < t' \end{cases} \;. \tag{15}$$

Calculating the difference gives

$$\Delta\Big(\phi(x)\phi^\dagger(x')\Big) = \begin{cases} \Delta^{(+)}(x - x') \;, & t > t' \\ -\Delta^{(-)}(x - x') \;, & t < t' \end{cases} \;. \tag{16}$$

where

$$\begin{aligned} \Delta^{(\pm)}(x - x') &\equiv \Big[\phi^{(\pm)}(x), \phi^{(\pm)\dagger}(x')\Big] \\ &= \pm \sum_{\mathbf{k}} \frac{e^{\mp ik(x-x')}}{2\omega_{\mathbf{k}}} \;. \end{aligned} \tag{17}$$

The propagator is usually written as a Fourier integral[2],

$$\Delta\Big(\phi(x)\phi^\dagger(x')\Big) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-x')}}{k^2 - m^2 + i0} \;. \tag{18}$$

In this form it is easy to see that (i times) the scalar propagator is the Green's function of the Klein-Gordon equation.

Note that the propagator is not an operator, but a (complex) number. This is because the commutator of the (scalar) field operators is a number.

[1] the operators inside the interaction Lagrangian $\mathcal{L}_v(x)$ (that is, the fields corresponding to the same point x) are postulated to be already in the normal order.

[2] where $\sum_{\mathbf{k}} \overset{V\to\infty}{\longrightarrow} \int \frac{d^3k}{(2\pi)^3}$.

For fermionic fields the T- and N-products have to be redefined slightly such that their difference, the propagator, is expressed through the anti-commutator. Specifically,

$$T\left(\psi_\alpha(x)\bar{\psi}_\beta(x')\right) = \begin{cases} \psi_\alpha(x)\bar{\psi}_\beta(x')\,, & t > t' \\ -\bar{\psi}_\beta(x')\psi_\alpha(x)\,, & t < t' \end{cases}\,, \tag{19}$$

and

$$\begin{aligned} N\left(\psi_\alpha(x)\bar{\psi}_\beta(x')\right) = & -\bar{\psi}^{(+)}_\beta(x')\psi^{(+)}_\alpha(x) \\ & +\psi^{(+)}_\alpha(x)\bar{\psi}^{(-)}_\beta(x') \\ & +\psi^{(-)}_\alpha(x)\bar{\psi}^{(+)}_\beta(x') \\ & +\psi^{(-)}_\alpha(x)\bar{\psi}^{(-)}_\beta(x')\,, \end{aligned} \tag{20}$$

where α and β are the indices in the bispinor space.

Analogously, (i times) the fermionic propagator must be the Green's function of the Dirac equation[3],

$$\Delta\left(\psi_\alpha(x)\bar{\psi}_\beta(x')\right) = \int \frac{d^4p}{(2\pi)^4}\left(\frac{i}{\gamma p - m + i0}\right)_{\alpha\beta} e^{-ip(x-x')}\,. \tag{21}$$

For the electromagnetic field the propagator depends on the gauge. One particular choice (Feynman gauge) is

$$\Delta\left(A_a(x)A_b(x')\right) = \int \frac{d^4k}{(2\pi)^4}\frac{-i\eta_{ab}}{k^2+i0}e^{-ikx}\,, \tag{22}$$

where η_{ab} is the Minkowski metric tensor.

Wick's theorem

The T-product of fields is equal the sum of normal products of all possible combinations of propagators,

$$\begin{aligned} T(A\dots Z) = & N(A\dots Z) \\ & +\Delta(AB)N(B\dots Z) \\ & +\Delta(AB)\Delta(CD)N(E\dots Z) \\ & \pm(\text{all other combinations of propagators})\dots \end{aligned} \tag{23}$$

where $\pm$ indicates that when two anti-commuting operators are commuted (to move an operator next to its propagator partner) a minus sign should be recorded.

For example, for two fields

$$T\phi(x)\phi^\dagger(x') = N\phi(x)\phi^\dagger(x') + \Delta(\phi(x)\phi^\dagger(x'))\,. \tag{24}$$

[3] here the expression $\frac{1}{\gamma p - m}$ sigifies the inverse matrix $(\gamma p - m)^{-1}$.

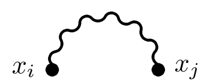
Feynman diagrams

A Feynman diagram is a graphical representation of a term in the Wick's expansion (23) of a perturbative term of the S-matrix (4).

Taking the interaction Lagrangian $\mathcal{L}_v = -g\bar{\psi}\psi\phi$ as an indicative example, the diagrams in coordinate space are drawn according to the following rules.

1. Each integration coordinate x_j is represented by a point:

2. The bosonic propagator $\Delta(\phi(x_i)\phi^\dagger(x_j))$ is represented by a bosonic (usually wiggly) line connecting points x_i and x_j:

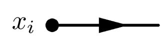


3. The fermionic propagator $\Delta(\psi(x_i)\bar{\psi}(x_j))$ is represented by a fermionic (usually solid) line connecting points x_i and x_j with an arrow from x_j to x_i;

4. A field $\phi(x_i)$ is represented by a bosonic line attached to the point x_i;

5. A field $\psi(x_i)$ is represented by a fermionic line attached to the point x_i with an arrow toward the point;

6. A field $\bar{\psi}(x_i)$ is represented by a fermionic line attached to the point x_i with an arrow from the point;



In practice Feynman diagrams are calculated in momentum space.

Examples of Feynman diagrams

For the interaction Lagrangian $\mathcal{L}_v = -g\bar{\psi}\psi\phi$ the second order term of the S-matrix is given as

$$S^{(2)} = \frac{(ig)^2}{2!} \int d^4x_1 d^4x_2$$
$$\times T\bar{\psi}(x_1)\psi(x_1)\phi(x_1)\bar{\psi}(x_2)\psi(x_2)\phi(x_2) . \quad (25)$$

The Wick's expansion of the T-product in (25) gives, among others, the term

$$N\bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2)\Delta(\phi(x_1)\phi(x_2)) , \quad (26)$$

According to Feynman rules the corresponding diagram, see figure 1, has two vertexes, x_1 and x_2, connected by a bosonic contraction $\Delta(\phi(x_1)\phi(x_2))$, and non-contracted fermionic lines $\bar{\psi}(x_1)$, $\psi(x_1)$ and $\bar{\psi}(x_2)$, $\psi(x_2)$ attached correspondingly to vertexes x_1 and x_2.

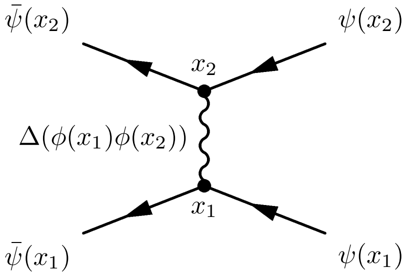


Figure 1: Feynman diagram representing the term (26).

Another interesting term in the Wick's expansion of the T-product in (25) is

$$\Delta(\psi(x_1)\bar{\psi}(x_2))N\bar{\psi}(x_1)\psi(x_2))\phi(x_1)\phi(x_2) . \quad (27)$$

Its Feynman diagram, see figure 2, has two vertexes, connected by the fermionic contraction $\Delta(\psi(x_1)\bar{\psi}(x_2))$ and the non-contracted fields $\bar{\psi}(x_1)$, $\phi(x_1)$ (with vertex x_1) and $\psi(x_2)$, $\phi(x_2)$ (with vertex x_2).

Initial and final states for Feynman diagrams

An non-contracted field ψ gives a non-zero result when acting on a particle state $a^\dagger_{\mathbf{p}\lambda}|0\rangle$ to the right,

$$\psi(x)a^\dagger_{\mathbf{p}\lambda}|0\rangle = u_{\mathbf{p}\lambda}e^{-ikx}|0\rangle \quad (28)$$

or on the anti-particle $b^\dagger_{\mathbf{k}\lambda}|0\rangle$ state to the left,

$$\langle 0|b_{\mathbf{p}\lambda}\phi(x) = \langle 0|v_{\mathbf{p}\lambda}e^{+ikx} \quad (29)$$

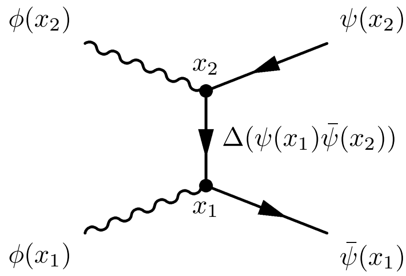


Figure 2: Feynman diagram representing the term (27).

An non-contracted field $\bar{\psi}$ acts non-vanishingly on an anti-particle to the right,

$$\bar{\psi}(x)b^\dagger_{\mathbf{p}\lambda}|0\rangle = \bar{v}_{\mathbf{p}\lambda}e^{-ipx}|0\rangle \quad (30)$$

or a particle to the left,

$$\langle 0|a_{\mathbf{p}\lambda}\bar{\psi}(x) = \langle 0|\bar{u}_{\mathbf{p}\lambda}e^{+ipx} \quad (31)$$

Thus,

1. The non-contracted field ψ can represent a particle in the initial state or an antiparticle in the final state.
2. The barred field $\bar{\psi}$ can represent an antiparticle in the initial state or a particle in the final state.
3. The non-contracted real field ϕ can represent a particle in both the initial and or final states.

If the initial state is assumed to be at the right side of the diagram and the final state at the left side, then figure 1 describes (a contribution to) fermion-fermion scattering, figure 2 – fermion-fermion annihilation into two bosons.

If the initial state is assumed at the bottom of the diagram and the final state at the top, then figure 1 describes (a contribution to) fermion-antifermion scattering, figure 2 – boson-fermion scattering.