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**Exercises**

1. Consider two interacting fields, a complex spin- $\frac{1}{2}$  field  $\psi$  and a real scalar field  $\phi$  with interacting Lagrangian  $\mathcal{L}_v = -g\bar{\psi}\psi\phi$ . Find the Euler-Lagrange equations and the Hamiltonian density  $T_0^0$  of the system.
2. For a real scalar field  $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$  show that commutation relations  $[a_k, a_k^\dagger] = 1$  for the generation/annihilation operators lead to the commutation relation  $[\chi, \pi] = i$  for the canonical generalized coordinate  $\chi$  and the corresponding generalized momentum  $\pi$ . Hints:

- (a) The generalized coordinates of a field are the values of the field itself,  $\chi = \phi$ . According to the general rule the generalized momentum is  $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\chi}}$ . Find the expression for  $\pi$ . (Answer:  $\pi = \partial_0\phi$ ).
- (b) Consider the amplitude of a single plane wave as the generalized coordinate (now a quantum-mechanical operator)

$$\chi = \phi_k = \frac{1}{\sqrt{2\omega_k}} \left( a_k e^{-ikx} + a_k^\dagger e^{ikx} \right)$$

and calculate the corresponding operator  $\pi$ .

- (c) Calculate the commutator  $[\chi, \pi]$ .
3. Time evolution in Heisenberg-Born-Jordan matrix mechanics: show that the canonical Hamilton equations of motion

$$\frac{\partial p}{\partial t} = -\frac{\partial H}{\partial q}, \quad \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p}, \quad \frac{\partial f(q, p)}{\partial t} = \{f, H\},$$

(where

$$\{f, H\} \equiv \frac{\partial f}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial H}{\partial q}$$

is the classical Poisson bracket) with the commutation relation  $[q, p] = i$  lead to operator equations

$$\frac{\partial p}{\partial t} = \frac{1}{i}[p, H], \quad \frac{\partial q}{\partial t} = \frac{1}{i}[q, H], \quad \frac{\partial f}{\partial t} = \frac{1}{i}[f, H].$$

Hint: show (by induction) that

$$\frac{1}{i}[q, f] = \frac{\partial f}{\partial p}, \quad \frac{1}{i}[p, f] = -\frac{\partial f}{\partial q}$$

for any function which can be represented as a series of powers of  $q$  and  $p$ .