

**Exercises: special relativity**

1. Argue that coordinate transformations between inertial frames of reference form a mathematical group.
2. Derive the Lorentz transformation, particularly, the velocity boost in  $x$ -direction,

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \begin{bmatrix} 1 & \frac{-v}{c^2} \\ -v & 1 \end{bmatrix} \begin{pmatrix} t \\ x \end{pmatrix},$$

using group postulates and the *locality of interactions* (in other words, the existence of a maximal speed of information transmission).

3. Show that in the non-relativistic limit,  $v \ll c$ , Lorentz transformations reduce to Galilean transformations.
4. Consider a composition of two Galilean transformations and derive the classical velocity-addition formula.
5. Consider a composition of two Lorentz transformations and derive the relativistic velocity-addition formula.
6. Formulate the relativistic velocity-addition formula for the *rapidity*  $\phi$  defined as

$$\tanh(\phi) = \frac{v}{c}.$$

Hint: use the law of addition of hyperbolic tangents,

$$\tanh(a + b) = \frac{\tanh(a) + \tanh(b)}{1 + \tanh(a)\tanh(b)}.$$

7. Show that the *infinitesimal interval*

$$ds^2 = c^2 dt^2 - d\mathbf{r}^2.$$

is invariant under a general Lorentz transformation (that is, boosts and spatial rotations). Show that the interval can also be written as

$$ds^2 = dx_a dx^a \equiv dx^a dx^b g_{ab}.$$

Show that in Minkowski space the finite interval,

$$\Delta s^2 = c^2 \Delta t^2 - \Delta \mathbf{r}^2.$$

is also invariant.

8. Show that the group of Lorentz transformations is  $SO(1, 3)$ .

Non-obligatory:

1. Show that a moving clock runs slower, than stationary. Hint: consider the transformation of

$$\begin{pmatrix} dt \\ dx = 0 \end{pmatrix}.$$

2. Show that a moving rod is shorter, than stationary. Hint: consider a transformation into

$$\begin{pmatrix} dt' = 0 \\ dx' \end{pmatrix}.$$

3. Which of the following objects are covariant?

- Kronecker delta  $\delta_b^a \equiv \{a = b?1 : 0\}$ ;
- Lorentz transformation matrix  $\Lambda_b^a$ ;
- Metric tensor  $g_{ab}$