Exercises: free scalar field

1. For a scalar field prove the orthogonality of plane waves with periodic boundary condition.

2. For the Lagrangian

$$\mathcal{L} = \partial_a \phi^* \partial^a \phi - m^2 \phi^* \phi$$

find the expressions for the energy-momentum tensor, T^a_b, and the conserved current density, j^a.

3. Calculate the 4-momentum $P^a = \int dV T^{0a}$ and the conserved current $J^a = \int_V dV j^a$ for positive and negative frequency normal modes. Interpret the results.

4. Prove that different normal modes contribute to the total energy and the total charge of a scalar field independently (that is, there are no interference terms in the sums).

5. Given $aa^\dagger - a^\dagger a = 1$ ($aa^\dagger + a^\dagger a = 1$) find the eigenvalues of the $a^\dagger a$ operator.