

**Exercises**

1. Prove that generators of a Lie group have a Lie algebra

$$[I_l, I_m] \doteq I_l I_m - I_m I_l = i C_{lm}^k I_k,$$

where  $C_{lm}^k$  are (structure) constants.

Hints:

- Consider the equation

$$g(\beta') = g(\alpha)g(\beta)g^{-1}(\alpha)$$

in the limit  $\beta \rightarrow 0$  and  $\alpha \rightarrow 0$ .

- Argue that

$$\lim_{\beta \rightarrow 0} \beta'_k = f_{kl}(\alpha) \beta_l,$$

where  $f_{kl}(\alpha)$  are some functions.

- Argue that

$$\lim_{\alpha \rightarrow 0} f_{kl}(\alpha) = \delta_{kl} + \sum_m C_{lm}^k \alpha_m,$$

where  $C_{lm}^k$  are some constants.

- Now consider the infinitesimal form of the equation.

2. Show that a representation of a Lie group is also a Lie group with the same Lie algebra, as the original group.
3. Show that if the operators

$$U = \exp\left(-i \sum_k J_k \alpha_k\right)$$

are unitary:  $U^\dagger U = 1$ , and the parameters  $\alpha_k$  are real, then the generators  $J_k$  are hermitian:  $J_k^\dagger = J_k$ . Show that if  $\det(U) = 1$  then  $\text{trace}(J_k) = 0$ .

Hints:

- consider only infinitesimal operators;
- show that  $\det(e^A) = e^{\text{trace}(A)}$  by diagonalizing matrix  $A$ .

4. Using the Lie algebra of the rotation group<sup>1</sup>

$$[I_j, I_k] = i \sum_l \epsilon_{jkl} I_l$$

find the 2x2 representation of the rotation generators  $\vec{I}$  (assuming  $I_3$  is diagonal and  $I_1$  is real).

Hints: recall that  $\text{trace}(I_k) = 0$  and  $I_k^\dagger = I_k$ ;  $I$ 's must be half the Pauli's  $\sigma$ -matrices.

<sup>1</sup> $\epsilon_{jkl}$  is the Levi-Civita (absolutely antisymmetric) tensor