Exercises: spin-$\frac{1}{2}$

1. Show that after a rotation by 2π a spinor changes sign.
   Hint: a 'spinor' is an object which under rotations of coordinates is transformed by a $j = \frac{1}{2}$ representation of the rotation group.

2. Show that the Lagrangian

   $$\mathcal{L} = \frac{1}{2}\left(\bar{\psi}\gamma^a\partial_a\psi + \partial_a\bar{\psi}\gamma^a\psi\right) - m\bar{\psi}\psi$$

   does not lead to a good Euler-Lagrange equation.

3. Show by direct calculation that the current $\bar{\psi}\gamma^a\psi$ conserves if ψ is a solution of the Dirac equation.

4. Calculate the energy-momentum tensor for the spin-$\frac{1}{2}$ field.

5. Show that if bispinor ψ is a solution to the Dirac equation, then its components satisfy the Klein-Gordon equation.
   Hint: multiply the Dirac equation by $(i\gamma^b\partial_b + m)$ from the left.

6. Argue that the 2×2 representation of the Lie algebra of the rotation group is the group of Special Unitary 2×2 matrices (called $SU(2)$). In other words, argue that $SU(2)$ group has the same Lie algebra, as the rotation group $SO(3)$.