Exercises: Lagrangian field theory

1. Derive the Newton's second law of motion for a particle with mass m moving in the potential $V(\mathbf{r})$,
$$m\frac{d\mathbf{v}}{dt} = -\nabla V(\mathbf{r}) \ ,$$
from the action
$$S = \int_{t_1}^{t_2} dt \left(\frac{m\mathbf{v}^2}{2} - V(\mathbf{r}) \right) \ .$$
Find also the particle's momentum $\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}}$ and energy $E = \mathbf{p}\cdot\mathbf{v} - L$.

2. Derive the famous expression,
$$E = mc^2 \ ,$$
for the rest-energy of a particle with mass m from the action
$$S = -mc \int ds$$
where the integral is taken along the trajectory of the particle.
Show that $E^2 = m^2c^4 + p^2c^2$.
Hints: show that the Lagrangian is equal
$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}} \ .$$

3. Derive the Klein-Gordon equation
$$\partial_a\partial^a\phi + m^2\phi = 0 \ ,$$
from the Lagrangian
$$\mathcal{L} = \partial_a\phi^*\partial^a\phi - m^2\phi^*\phi \ .$$

4. Derive the Maxwell equations with sources,
$$\partial_a\partial^a A^b = 4\pi j^b \ ,$$
from the Lagrangian
$$L = -\frac{1}{8\pi}\partial_a A^b \partial^a A_b - j^a A_a \ ,$$
with the Lorenz condition $\partial_a A^a = 0$.

Non-obligatory

1. Show that d^4x and dVj^0 are Lorentz scalars, and that dVT^{0a} is a Lorentz 4-vector.