CPT theorem

The *CPT theorem* states that in a canonical quantum field theory the action is invariant under the combination of charge conjugation (C), parity transformation (P), and time reversal (T).

The CPT theorem entails certain relations between physical observables, in particular equal masses and equal life-times for particles and antiparticles. At present, CPT is the sole combination of C, P, T observed as an exact symmetry of nature at the fundamental level

Let us see how it works using the complex scalar field as an example.

C: charge conjugation

Under charge conjugation the particles are exchanged with antiparticles,

$$a_{\mathbf{k}} \xrightarrow{C} b_{\mathbf{k}} \; . \tag{1}$$

The field $\phi(x)$ then turns into $\phi^\dagger(x)$,

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ikx} + b^\dagger_{\mathbf{k}} e^{+ikx} \right) \xrightarrow{C} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(b_{\mathbf{k}} e^{-ikx} + a^\dagger_{\mathbf{k}} e^{+ikx} \right) = \phi^\dagger(x) \; . \tag{2}$$

P: parity transformation

Under parity transformation the momentum changes sign,

$$a_{\mathbf{k}} \xrightarrow{P} a_{-\mathbf{k}} \; . \tag{3}$$

The field $\phi(t, \mathbf{x})$ then turns into $\phi(t, -\mathbf{x})$,

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ikx} + b^\dagger_{\mathbf{k}} e^{+ikx} \right) \xrightarrow{P} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{-\mathbf{k}} e^{-ikx} + b^\dagger_{-\mathbf{k}} e^{+ikx} \right) = \phi(t, -\mathbf{x}) \; . \tag{4}$$

T: time reversal

Under time reversal the process of annihilation of a particle turns into the process of generation of a particle with opposite momentum,

$$a_{\mathbf{k}} \xrightarrow{T} a^\dagger_{-\mathbf{k}} \; . \tag{5}$$

The field $\phi(t, \mathbf{x})$ then turns into $\phi^\dagger(-t, \mathbf{x})$,

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ikx} + b^\dagger_{\mathbf{k}} e^{+ikx} \right) \xrightarrow{T} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a^\dagger_{-\mathbf{k}} e^{-ikx} + b_{-\mathbf{k}} e^{+ikx} \right) = \phi^\dagger(-t, \mathbf{x}) \; . \tag{6}$$

Combined transformation: CPT

Under the combination of all three the field ϕ changes the sign of the argument,

$$\phi(x) \xrightarrow{CPT} \phi(-x) \; . \tag{7}$$

However, since the action involves integration $\int d^4x$ over the whole space, the sign change of the argument of a field leaves the action unchanged, q.e.d.

Gauge theories

Gauge theory is a peculiar quantum field theory where the Lagrangian is invariant under continuous group of local transformations, called *gauge transformations*. The transformations form a Lie group which is referred to as the *symmetry group* or *gauge group* of the theory.

For each group parameter there is a special vector field, called *gauge field*, which ensures the invariance of the Lagrangian under the gauge transformation. The quanta of the gauge field are called *gauge bosons*.

If the symmetry group is commutative, the gauge theory is called *abelian*, otherwise it is called *non-abelian* or *Yang-Mills* theory.

Historically the quantum electro-dynamics (QED) was first recognised as a gauge theory with the gauge group $U(1)$. Then it turned out that only gauge theories can be renormalized and therefore the Standard Model had to be built as a gauge theory, though in order to incorporate more gauge bosons the gauge group had to be enlarged. It turned out that the groups $SU(2)$ and $SU(3)$ fit perfectly for the weak and strong interactions correspondingly.

Quantum electro-dynamics

Quantum electrodynamics (QED) is a theory of the electron/positron (bispinor) field ψ coupled to the electromagnetic (vector) field A_a with the interaction Lagrangian $-j_a A^a$, where $j_a = g\bar{\psi}\gamma_a\psi$ is the conserved current and g is the charge of electron.

The QED Lagrangian,

$$\mathcal{L}_{\mathrm{QED}} = \bar{\psi}\gamma^a i(\partial_a + igA_a)\psi - m\bar{\psi}\psi - \frac{1}{4}F_{ab}F^{ab}, \tag{8}$$

is invariant under the local gauge transformation,

$$\left\{ \begin{array}{lcl} \psi & \to & \psi' = e^{ig\alpha(x)}\psi \\ A_a & \to & A'_a = A_a - \partial_a \alpha(x) \end{array} \right. . \tag{9}$$

where $\alpha(x)$ is an arbitrary scalar function (and also the group parameter). The transformation matrices $e^{ig\alpha}$ form a Lie group $U(1)$ where the charge is the group generator. QED is thus a gauge theory with the symmetry group $U(1)$. The group is commutative, therefore QED is an abelian theory.

The QED Lagrangian (8) can be conveniently written as[1]

$$\mathcal{L}_{\rm QED} = i\bar\psi\gamma^a D_a\psi - m\bar\psi\psi - \frac{1}{4}F^2 \,, \tag{10}$$

where the *group covariant derivative*,

$$D_a = \partial_a + igA_a \,, \tag{11}$$

has the property that under the gauge transformation

$$D_a\psi \to e^{ig\alpha(x)} D_a\psi \,. \tag{12}$$

The gauge field tensor can be written through group covariant derivatives as

$$F_{ab} = \frac{1}{ig}[D_a, D_b] \,. \tag{13}$$

Non-abelian (Yang-Mills) gauge theories

The QED with the $U(1)$ symmetry group has one gauge boson, the photon, while more gauge bosons are needed for the standard model. Yang and Mills suggested to consider a more general group of matrices,

$$U(x) = e^{iI_j\alpha_j(x)} \tag{14}$$

with generators I_j, $j = 1 \dots n$, and Lie-algebra

$$[I_j, I_k] = C^l_{jk} I_l. \tag{15}$$

If the structure constants C^l_{jk} are not equal zero, the theory is called non-abelian.

To make the theory gauge invariant a separate vector field A^a_j is needed for each group parameter α_j and the gauge field becomes $A^a = I^j A^a_j$ (there is no difference whether the group index j is up or down). The generalized gauge transformation is now defined as

$$\left\{ \begin{array}{ccl} \psi & \to & \psi' = U\psi \,, \\ A_a & \to & A'_a = UA_aU^{-1} - \frac{1}{ig}\partial_a U \cdot U^{-1} \,. \end{array} \right. \tag{16}$$

The transformation matrices U are of dimension n and thus the field ψ, on which the matrix operates, becomes a column-vector in this new dimension[2].

In a Yang-Mills theory the fermionic field ψ is then a rather non-trivial object: it is a generation/annihilation operator in the space of quantum states of the field; a Lorentz group bispinor; and an object rotated by a (usually fundamental) representation of the gauge group.

The Yang-Mills guage-field tensor is

$$\begin{aligned} F_{ab} &= \frac{1}{ig}[D_a, D_b] \\ &= \partial_a A_b - \partial_b A_a + ig[A_a, A_b] \\ &= I_j F^j_{ab} \,. \end{aligned} \tag{17}$$

The Yang-Mills Lagrangian for the gauge field is

$$\mathcal{L}_{\rm YM} = -\frac{1}{2}{\rm Trace}(F^2) = -\frac{1}{4}F^{ab}_j F^j_{ab} \,. \tag{18}$$

with the standard normalization

$${\rm Trace}(I_j I_k) = \tfrac{1}{2}\delta_{jk} \,. \tag{19}$$

Since the Yang-Mills field tensor contains the gauge field commutator (which is of second order in the field) the Yang-Mills Lagrangian contains third and fourth order terms. The non-abelian gauge bosons can thus self-interact.

The gauge bosons are necessarily massless (as the mass term breaks gauge invariance). However with the so called Higgs mechanism the gauge bosons can acquire effective mass through interactions with the Higgs field.

The number of gauge bosons is equal to the number of generators in the gauge group. There is one gauge boson, the photon, for the $U(1)$ group; three gauge bosons for the $SU(2)$ group; and eight gauge bosons for the $SU(3)$ group.

Exercises

1. Find the infinitesimal gauge-transformation rule for the gauge field A^a_j.

2. Show that the Yang-Mills field tensor (17) is gauge invariant. Hint: see Mulders, section 12.1.

[1] $F^2 \equiv F^{ab}F_{ab}$

[2] this dimension is referred to as *weak isospin space* for the weak interaction and *color space* for the strong interaction.