Massive spin-1 field

The lowest dimension Lorentz group representation, which contains spin-1, is $(\frac{1}{2}, \frac{1}{2})$. It transforms four-vectors $\varphi^\mu = \{\varphi^0, \vec{\varphi}\}$[1].

There is a problem though, namely that a four-vector also contains a spin-0 part, the rotation scalar. The extra component can be excluded by imposing some additional condition, for example, the Lorentz condition

$$\partial_\mu \varphi^\mu = 0 \ . \tag{1}$$

For a plane wave $\varphi^\mu = \epsilon^\mu e^{-ipx}$, where ϵ^μ is a four-vector and $p_\mu p^\mu = m^2$, the Lorentz condition gives $\epsilon p = 0$. In the rest frame ($\mathbf{p} = 0$) the latter leads to $\epsilon^0 = 0$ indicating that only three components of the vector $\vec{\epsilon}$ are independent, which is consistent with the concept of a spin-1 field.

Lagrangian

The suitable Lagrangian is

$$\mathcal{L} = -\partial_\mu \varphi^*_\nu \partial^\mu \varphi^\nu + m^2 \varphi^*_\nu \varphi^\nu \ . \tag{2}$$

It looks very much like a sum of several Lagrangians for spin-0 field.

Euler-Lagrange equation

Each component satisfies the Klein-Gordon equation,

$$\left(\partial_\mu \partial^\mu + m^2\right) \varphi^\nu = 0 \ . \tag{3}$$

Normal modes

$$\varphi = \sum_{\mathbf{k}\lambda} \frac{1}{\sqrt{2\omega_\mathbf{k}}} \left(a_{\mathbf{k}\lambda} \epsilon_\lambda e^{-ikx} + b^+_{\mathbf{k}\lambda} \epsilon^*_\lambda e^{+ikx}\right) \ , \tag{4}$$

where the spin functions ϵ_λ are chosen in the rest frame (where $(\epsilon_\lambda)^0 = 0$) as eigenfunction of the I_3 generator, $I_3 \epsilon_\lambda = \lambda \epsilon_\lambda$, $\lambda = 1, 0, -1$. They are normalized as $\epsilon^\dagger_\lambda \epsilon_{\lambda'} = \delta_{\lambda\lambda'}$.

The generation/annihilation operators satisfy bosonic commutation relations.

Electromagnetic field

Electromagnetic field is a massless field. It can be described by a four-vector potential A^μ, indicating that it is a spin-1 field. The Lagrangian of the electromagnetic field in the Gauss units is

$$\mathcal{L} = -\frac{1}{8\pi} \partial_\mu A^\nu \partial^\mu A_\nu \ . \tag{5}$$

Equivalently, the Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}; \ , \tag{6}$$

where $F^{\mu\nu}$ is the electromagnetic tensor,

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \ . \tag{7}$$

The absence of the mass term leads to an important symmetry, called *gauge symmetry*. The Lagrangian is invariant under a transformation

$$A^\mu \to A^\mu + \partial^\mu \phi \ , \tag{8}$$

called gauge transformation, where ϕ is an arbitrary scalar function of coordinates. The Lorentz condition limits the class of arbitrary function to harmonic functions

$$\partial^\mu \partial_\mu \phi = 0 \ . \tag{9}$$

The Euler-Lagrange equation for the components of the four-potential is the mass-zero Klein-Gordon equation,

$$\partial^\mu \partial_\mu A^\nu = 0 \ . \tag{10}$$

Let us look for the solutions in the form of a plane wave,

$$A(k) = \sqrt{\frac{2\pi}{\omega}} e(k) e^{-ikx} \ , \tag{11}$$

where $k = \{\omega, \mathbf{k}\}$, $\omega = |\mathbf{k}|$, $k^2 = 0$ (since photon mass is zero), and $e(k)$ is a four-vector.

From the Lorentz condition,

$$ke = 0 \ , \tag{12}$$

it follows that the amplitude $e(k)$ is not time-like $\Rightarrow$ $\mathbf{e} \neq 0$.

A gauge transformation with $\phi = ife^{-ikx}$, where f is a scalar, leads to a transformation of the amplitude

$$e_\mu \to e_\mu + fk_\mu \ . \tag{13}$$

The scalar f can always be chosen such that in a certain frame

$$e = \{0, \mathbf{e}\} \ , \ \ \mathbf{ke} = 0; \tag{14}$$

and

$$A_0 = 0 \ , \ \ \nabla \mathbf{A} = 0 \ . \tag{15}$$

The gauge (15) is called "Coulomb gauge" (also known as "transverse" or "radiation" gauge)

In the transverse gauge the electromagnetic field can be represented as

$$\mathbf{A} = \sum_{\mathbf{k}\lambda} \sqrt{\frac{2\pi}{\lambda}} \left(a_{\mathbf{k}\lambda} \mathbf{e}_\lambda e^{-ikx} + a^\dagger_{\mathbf{k}\lambda} \mathbf{e}^*_\lambda e^{ikx}\right) \ , \tag{16}$$

[1] Another low-dimension representation with spin-1 is the six-component object $(1,0) \oplus (0,1)$. However, these two objects will turn out to be related, as are four-component electromagnetic potential and six-component electromagnetic field.

where $\lambda = 1, 2$ are the two orthogonal polarizations,

$$\mathbf{e}_\lambda \mathbf{e}_{\lambda'} = \delta_{\lambda\lambda'} \ , \quad \mathbf{ek} = 0 \ . \tag{17}$$

and the generation/annihilation operators $a_{\mathbf{k}\lambda}$, $a^\dagger_{\mathbf{k}\lambda}$ satisfy the bosonic commutaion relations

$$\left[a_{\mathbf{k}\lambda}, a^\dagger_{\mathbf{k}\lambda}\right] = \delta_{\lambda\lambda'}\delta_{\mathbf{kk}'} \ . \tag{18}$$

Spin-statistics theorem

Canonical quantum field theory predicts (with certain caveats) that integer spin fields are bosonic, half-integer spin fields are fermionic.

Exercises

1. Show that the Lagrangians

$$\mathcal{L} = -\frac{1}{8\pi}\partial_\mu A^\nu \partial^\mu A_\nu$$

   and

$$\mathcal{L} = -\frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}$$

   are equivalent under the Lorentz condition $\partial_\mu A^\mu = 0$. Hint: the difference is a full derivative, which does not contribute to the variation of the action.

2. Show that the Lagrangian

$$\mathcal{L} = -\frac{1}{8\pi}\partial_\mu A^\nu \partial^\mu A_\nu - j^\mu A_\mu$$

   is gauge invariant if j^μ is a conserved current ($\partial_\mu j^\mu = 0$).

3. Derive the Maxwell equation with sources from the Lagrangian

$$\mathcal{L} = -\frac{1}{8\pi}\partial_\mu A^\nu \partial^\mu A_\nu - j^\mu A_\mu$$

4. Derive the expression $q\left(\mathbf{E} + \mathbf{v} \times \mathbf{H}\right)$ for the Lorentz force (the force acting on a charged point particle due to the electromagnetic field) from the action

$$S = -m \int ds - q \int dx^\mu A_\mu \ ,$$

   where the integral is take along the trajectory of the particle, $ds^2 = dt^2 - d\mathbf{r}^2$, m and q are the mass and the charge of the particle, A_μ is the electromagnetic field at the point where the particle is located.