# Spin-1/2 field: Canonical quantization

## Euler-Lagrange equation, current, energy

The simplest covariant Lagrangian for a spin-1/2 field is given as (the real part of)

$$\mathcal{L} = i\bar{\psi}\gamma_\mu\partial^\mu\psi - m\bar{\psi}\psi \ , \tag{1}$$

the Euler-Lagrange equation of which is called the Dirac equation,

$$(i\gamma_\mu\partial^\mu - m)\psi = 0 \ . \tag{2}$$

The conserved current is given as

$$j^\mu = \bar{\psi}\gamma^\mu\psi \ , \tag{3}$$

and the energy density as

$$T^0_0 = -i\bar{\psi}\vec{\gamma}\vec{\partial}\psi + m\bar{\psi}\psi \ . \tag{4}$$

## Plane-wave normal modes

Let us look for a solution in the form of a plane waves,

$$\psi(x) = \begin{pmatrix}\phi\\ \chi\end{pmatrix} e^{-ipx} \equiv \begin{pmatrix}\phi\\ \chi\end{pmatrix} e^{i\vec{p}\vec{r}-iEt} \ . \tag{5}$$

Substituting this into the Dirac equation gives (in the Dirac basis)

$$\begin{bmatrix} E-m & -\vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -(E+m)\end{bmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = 0 \ . \tag{6}$$

The homogeneous system of linear algebraic equations has a non-trivial solution only when the determinant of the matrix is zero, which gives the relativistic relation between $E$ and $\vec{p}$,

$$E^2 = m^2 + \vec{p}^{\,2} \ . \tag{7}$$

This in turn leads the the existence of both positive- and negative-frequency solutions

$$u_{\vec{p}}\, e^{-ipx} \ , \text{ and } \ v_{\vec{p}}\, e^{+ipx} \ , \tag{8}$$

where $p^\mu = \{E_{\vec{p}}\ , \vec{p}\}$ and $E_{\vec{p}}$ is the positive square root $E_{\vec{p}} = +\sqrt{m^2+\vec{p}^{\,2}}$.

For the bispinors $u_{\vec{p}}$ and $v_{\vec{p}}$ we get from the Dirac equation,

$$u_{\vec{p}} = \begin{pmatrix}\phi \\ \frac{\vec{\sigma}\vec{p}}{E_{\vec{p}}+M}\phi\end{pmatrix} \ , \ v_{\vec{p}} = \begin{pmatrix}\frac{\vec{\sigma}\vec{p}}{E_{\vec{p}}+M}\chi \\ \chi\end{pmatrix} \tag{9}$$

where $\phi$ and $\chi$ are arbitrary spinors.

There are two linearly independent spinors, which can be chosen, for example, as eigen-spinors of the $\frac{1}{2}\sigma_3$ operator in the frame where $\vec{p}$=0,

$$\frac{1}{2}\sigma_3\phi_\lambda = \lambda\phi_\lambda \ , \tag{10}$$

and similarly for the spinor $\chi$.

## Normalization to unit charge

The charge density for the spin-1/2 field is

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi \ . \tag{11}$$

Normalization to unit charge $Q = \int_{V=1} d^3x j^0$ then gives for our plane-waves

$$Q\left[ue^{-ipx}\right] = u^\dagger u = \phi^\dagger\phi\frac{2E_{\vec{p}}}{E_{\vec{p}}+m} \to 1 \ , \tag{12}$$

$$Q\left[ve^{+ipx}\right] = v^\dagger v = \chi^\dagger\chi\frac{2E_{\vec{p}}}{E_{\vec{p}}+m} \to 1 \ (!). \tag{13}$$

Note that unlike the scalar field the spin-1/2 field seemingly gives positive charges for both positive- and negative-frequency solutions.

Unit charge condition for a plane-wave leads to the normalizations

$$u^\dagger_\lambda u_{\lambda'} = v^\dagger_\lambda v_{\lambda'} = \delta_{\lambda\lambda'} \ , \tag{14}$$

which are obtained by choosing

$$\phi^\dagger_\lambda\phi_{\lambda'} = \chi^\dagger_\lambda\chi_{\lambda'} = \frac{E_{\vec{p}}+m}{2E_{\vec{p}}}\delta_{\lambda\lambda'} \ . \tag{15}$$

With this normalization

$$\bar{u}_\lambda u_{\lambda'} = -\bar{v}_\lambda v_{\lambda'} = \frac{m}{E_{\vec{p}}}\delta_{\lambda\lambda'} \ . \tag{16}$$

## Energies of the normal modes

From (4) we get that the positive-energy solution has positive energy,

$$E\left[ue^{-ipx}\right] = \bar{u}\vec{\gamma}\vec{p}u + m\bar{u}u = E_{\vec{p}} \ , \tag{17}$$

while the negative-energy solution seemingly has negative energy,

$$E\left[ve^{+ipx}\right] = \bar{v}\vec{\gamma}(-\vec{p})v + m\bar{v}v = -E_{\vec{p}} \ . \ (!) \tag{18}$$

Apparently out theory indicates that spin-1/2 fields cannot exist as classical fields (which agrees with the experiment).

## Charge and Hamiltonian in the normal mode representation

An arbitrary solution to the Dirac equation can be represented as a linear combination of normal modes,

$$\psi = \sum_{\vec{p}\lambda}\left(a_{\vec{p}\lambda}u_{\vec{p}\lambda}e^{-ipx} + b^\dagger_{\vec{p}\lambda}v_{\vec{p}\lambda}e^{-ipx}\right) \ . \tag{19}$$

The charge of the field is then given as

$$Q=\sum_{\vec p\lambda}\left(a^\dagger_{\vec p\lambda}a_{\vec p\lambda}+b_{\vec p\lambda}b^\dagger_{\vec p\lambda}\right)\ , \tag{20}$$

and the Hamiltonian,

$$H=\sum_{\vec p\lambda}E_{\vec p}\left(a^\dagger_{\vec p\lambda}a_{\vec p\lambda}-b_{\vec p\lambda}b^\dagger_{\vec p\lambda}\right)\ . \tag{21}$$

**Anti-commutaion relations for generation/annihilation operators**

The only way to make sense of the charge and Hamiltonian is to postulate *anti-commutation* relation for the spin-1/2 generation/annihilation operators,

$$a_{\vec p\lambda}a^\dagger_{\vec p\,'\lambda'}+a^\dagger_{\vec p\,'\lambda'}a_{\vec p\lambda}=\delta_{\vec p\vec p\,'}\delta_{\lambda\lambda'}\ , \tag{22}$$

$$b_{\vec p\lambda}b^\dagger_{\vec p\,'\lambda'}+b^\dagger_{\vec p\,'\lambda'}b_{\vec p\lambda}=\delta_{\vec p\vec p\,'}\delta_{\lambda\lambda'}\ . \tag{23}$$

Thus in canonical quantum field theory spin-1/2 particles are necessarily fermions.

With this postulates the charge and the energy take the form we wanted,

$$Q=\sum_{\vec p\lambda}\left(n_{\vec p\lambda}-\bar n_{\vec p\lambda}\right)\ , \tag{24}$$

$$E=\sum_{\vec p\lambda}E_{\vec p}\left(n_{\vec p\lambda}+\bar n_{\vec p\lambda}\right)\ , \tag{25}$$

where

$$n_{\vec p\lambda}=a^\dagger_{\vec p\,'\lambda'}a_{\vec p\lambda}\ , \tag{26}$$

and

$$\bar n_{\vec p\lambda}=b^\dagger_{\vec p\,'\lambda'}b_{\vec p\lambda}\ , \tag{27}$$

are the number-of-particle and number-of-anti-particle operators with eigenvalues 0 and 1.

**Exercises**

1. Show that for a classical particle with the action

$$S=\int dtL(q,\dot q)$$

   (a) the Euler-Lagrange equation is

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial\dot q}=\frac{\partial L}{\partial q}\ .$$

   (b) the (translation invariance) momentum is

$$p=\frac{\partial L}{\partial\dot q}\ .$$

   (c) the (time invariance) energy is

$$E=\frac{\partial L}{\partial\dot q}\dot q-L\ .$$

2. Show that for a classical particle with mass $m$ the action

$$S=-m\int ds\ ,$$

   where $ds^2=dt^2-d\vec r^{\,2}$ leads to the relativistic relation between energy and momentum,

$$E^2=m^2+\vec p^{\,2}\ .$$

   Hints:

   (a) Show that the Lagrangian is

$$L=-m\sqrt{1-\vec v^{\,2}}.$$

   (b) Show that the momentum is

$$\vec p=\frac{m\vec v}{\sqrt{1-\vec v^{\,2}}}.$$

   (c) Show that the energy is

$$E=\frac{m}{\sqrt{1-\vec v^{\,2}}}.$$

3. Calculate the current and the momentum of the positive- and negative-frequency solutions of the Dirac equation.
4. Find the projection operators $\mathcal{P}_\pm$ on the positive and negative frequency solutions of the Dirac equation.
5. The spinors $\phi\in(\frac12,0)$ and $\chi\in(0,\frac12)$ are often called “left” and “right”. Find out why. Hint: find the projection of the spin on the velocity vector as function of the velocity with which the spinor moves relative to the observer: consider a state with equal z-projections of the spin and then boost it along the z-axis. What happens if the spinor has zero-mass and is thus doomed to forever move with the speed of light?
6. Show that $\frac{d^3p}{2E_{\vec p}}$ is invariant.