Spin-$\frac{1}{2}$ field: bispinors

Bispinors

The lowest irreducible representation, containing spin-$\frac{1}{2}$, of the covariance group $O(1,3)$ is $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$. The matrices from this representation transform four-component objects called bispinors

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} , \tag{1}$$

where $\phi \in (\frac{1}{2},0)$ is a (right) (Weyl) spinor with the infinitesimal transformation matrix

$$g = 1 + i\frac{1}{2}\vec{\sigma}d\vec{\omega} + i0d\vec{\omega}^{\star} , \tag{2}$$

and $\chi \in (0,\frac{1}{2})$ is a (left) (Weyl) spinor with the infinitesimal transformation matrix

$$1 + i0d\vec{\omega} + i\frac{1}{2}\vec{\sigma}d\vec{\omega}^{*} = (g^{\dagger})^{-1} . \tag{3}$$

Thus under the Lorentz transformation

$$\phi \to g\phi \, , \; \chi \to (g^{\dagger})^{-1}\chi . \tag{4}$$

Under the parity transformation $\mathcal{P}$ the Weyl spinors transform into each other,

$$\phi \stackrel{\mathcal{P}}{\to} \chi \, , \; \chi \stackrel{\mathcal{P}}{\to} \phi . \tag{5}$$

Bilinear forms of bispinors

From the transformation laws (4,5) it follows, that the direct product $\psi^{*} \otimes \psi$ can be reduced to five irreducible covariant objects:

1. scalar, $\bar{\psi}\psi$;
2. pseudo-scalar, $\bar{\psi}\gamma_5\psi$;
3. vector, $\bar{\psi}\gamma^{\mu}\psi$;
4. pseudo-vector, $\bar{\psi}\gamma^{\mu}\gamma_5\psi$;
5. antisymmetric tensor, $\bar{\psi}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})\psi$,

where $\bar{\psi} \equiv \psi^{\dagger}\gamma_0$ and the block-matrices γ^{μ} and γ_5, called the gamma matrices, are given (in the Weyl basis) as

$$\gamma_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{6}$$

$$\vec{\gamma} = \begin{bmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} \tag{7}$$

$$\gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{8}$$

A similarity transformation

$$\begin{aligned} \psi &\to S\psi \\ \gamma^{\mu} &\to S\gamma^{\mu}S^{-1} \end{aligned} \tag{9}$$

with the matrix

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{10}$$

changes Weyl basis into Dirac basis, where the γ-matrices are

$$\gamma_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{11}$$

$$\vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix} \tag{12}$$

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \tag{13}$$

The γ-matrices satisfy the anti-commutation relation

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} , \tag{14}$$

where $g^{\mu\nu}$ is the Minkowski metric tensor. The matrix γ_5 anti-commutes with γ^{μ},

$$\{\gamma^{\mu}, \gamma_5\} = 0 . \tag{15}$$

Lagrangian

The Lagrangian should be a real Lorentz scalar, bilinear in ψ and $\partial_{\mu}\psi$. A suitable form is

$$\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \partial_{\mu}\bar{\psi}\gamma^{\mu}\psi\right) - m\bar{\psi}\psi . \tag{16}$$

Euler-Lagrange equation: the Dirac equation

Using the general expression,

$$\partial_{\mu}\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\bar{\psi})} = \frac{\partial\mathcal{L}}{\partial\bar{\psi}} , \tag{17}$$

the Lagrangian (16) leads to the Dirac equation,

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0 . \tag{18}$$

Energy-momentum and conserved current

$$j^{\mu} = \bar{\psi}\gamma^{\mu}\psi \tag{19}$$

$$T_0^0 = -i\bar{\psi}\vec{\gamma}\vec{\partial}\psi + m\bar{\psi}\psi \tag{20}$$

Exercises

1. Show that the 2×2 representation of the Lie algebra[1] of the rotation group is the $SU(2)$ group, or, in other words, that the $SU(2)$ group has the same Lie algebra, as the rotation group $SO(3)$.

2. Show that after a rotation by 2π a spinor (an object, transformed by a 2×2 representation of the rotation group) changes sign.

3. Show that the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left(\bar{\psi}\gamma^\mu\partial_\mu\psi + \partial_\mu\bar{\psi}\gamma^\mu\psi\right) - m\bar{\psi}\psi$$

   does not lead to a good Euler-Lagrange equation.

4. Show by direct calculation that the current $\bar{\psi}\gamma^\mu\psi$ conserves if ψ is a solution of the Dirac equation.

5. Calculate the energy-momentum tensor for the spin-1/2 field.

6. Show that if a bispinor ψ is a solution to the Dirac equation, then its components satisfy the Klein-Gordon equation.

[1] in physics, a representation of a Lie algebra is a group of matrices whose generators have the given Lie algebra