Transformational properties of fields

It was relatively easy to build a covariant theory for a scalar (one-component) field. For multi-component fields (like the electro-magnetic field) we need to learn their transformational properties under coordinate transformations.

The group of coordinate transformations

The coordinate transformations $x \to x' = ax$ between inertial frames form a group: the group operation is composition, the identity element is identical transformation, and the inverse element is the inverse transformation.

An n-component physical quantity ψ transforms with some $n \times n$ matrices t_a, $\psi \to \psi' = t_a\psi$, which form a group homomorphic to the group of coordinate transformations, since the group operations are apparently preserved,

$$t_{a_1 a_2} = t_{a_1} t_{a_2},\ t_{a^{-1}} = (t_a)^{-1},\ t_{a=\mathbf{1}} = \mathbf{1}\ . \quad (1)$$

The group t_a is referred to as a *representation* of the group of coordinate transformation.

Lie groups and Lie algebras

For rotations and velocity boosts the transformation matrices are differentiable functions of the transformation parameters.

A group whose elements g are differentiable functions of a set of continuous parameters, $g(\alpha_1, ..., \alpha_n)$, is called a *Lie group.*

An element close to 1 (assuming $g(0) = 1$) can be expressed in terms of the *generators* I_k,

$$g(d\alpha) = 1 + i\sum_{k=1}^{n} I_k d\alpha_k \quad (2)$$

The commutation relations of the generators are called *Lie algebra,*

$$I_j I_m - I_m I_j = i\sum_{k=1}^{n} C^k_{jm} I_k, \quad (3)$$

where C^k_{jm} is the so called the *structure constant.* The Lie algebra largely defines the properties of a Lie group.

Lie algebra of the rotation group

The transformation of the coordinates under a rotation around z-axis over the angle θ is given as

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \quad (4)$$

For an infinitesimally small angle $d\theta$ the rotation matrix can be written as

$$1 + i \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\theta \equiv 1 + iI_z d\theta\ , \quad (5)$$

where I_z is the generator of an infinitesimal rotation around z-axis. The corresponding generators I_x and I_y are

$$I_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix},\ I_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} . \quad (6)$$

Direct calculation shows that these generators have the Lie algebra

$$I_k I_l - I_l I_k = i\sum_m \epsilon_{klm} I_m\ , \quad (7)$$

where ϵ_{klm} is an antisymmetric tensor of rank 3.

Rotation group is denoted SO(3), which stands for special (determinant 1) orthogonal matrix 3×3.

The Lie algebra of the Lorentz group

The Lorentz group consists of rotations and velocity boosts. The 4-dimensional rotation generators can be written immediately from (5) and (6) as

$$J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} . \quad (8)$$

The Lorentz boost matrix for an infinitesimally small relative velocity dv along one of the axes is

$$\begin{pmatrix} 1 & -dv \\ -dv & 1 \end{pmatrix} = 1 + i \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} dv\ . \quad (9)$$

Thus the three generators of velocity boosts are

$$K_z = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} K_y = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K_x = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (10)$$

Having the generators, one can calculate the Lie algebra of the Lorentz group

$$J_kJ_l - J_lJ_k = i\sum_m \epsilon_{klm}J_m \quad (11)$$

$$J_kK_l - K_lJ_k = i\sum_m \epsilon_{klm}K_m \quad (12)$$

$$K_kK_l - K_lK_k = -i\sum_m \epsilon_{klm}J_m\ . \quad (13)$$

The infinitesimal group element corresponding to a rotation around the direction $\vec{n}$ over an angle $d\theta$ and a velocity boost $d\vec{v}$ is given as[1]

$$g = 1 + i\vec{J}\vec{n}d\theta + i\vec{K}d\vec{v}. \quad (14)$$

The Lie algebra of the Lorentz group can be written in a more symmetric way with a (complex) parametrization

$$d\vec{w} = \vec{n}d\theta + id\vec{v}. \quad (15)$$

The infinitesimal Lorentz transformation is then

$$g = 1 + i\vec{M}d\vec{w} + i\vec{N}d\vec{w}^*\ , \quad (16)$$

where the (hermitian) generators $\vec{M}$ and $\vec{N}$ are linear combinations of generators $\vec{J}$ and $\vec{K}$

$$\vec{M} = \frac{1}{2}(\vec{J} - i\vec{K})\ ,\ \vec{N} = \frac{1}{2}(\vec{J} + i\vec{K})\ . \quad (17)$$

The Lie algebra for the new generators is

$$M_kM_l - M_lM_k = i\sum_m \epsilon_{klm}M_m \quad (18)$$

$$N_kN_l - N_lN_k = i\sum_m \epsilon_{klm}N_m \quad (19)$$

$$M_kN_l - N_lM_k = 0. \quad (20)$$

Thus the Lorentz Lie algebra is a combination of two independent rotation Lie algebras.

Irreducible representations of the rotation group

Canonically, instead of the generators I_x, I_y, I_z we shall use another parametrization, I_+, I_-, I_z, where

$$I_\pm = \frac{1}{\sqrt{2}}(I_x \pm I_y), \quad (21)$$

with the commutation relation

$$I_zI_\pm - I_\pm I_z = \pm I_\pm\ ,\ I_+I_- - I_-I_+ = I_z. \quad (22)$$

[1] where $\vec{a}\vec{b} \equiv a_1b_1 + a_2b_2 + a_3b_3$

Let us look for the representations where the I_z matrix is diagonal. Let it have eigenvectors $|\lambda\rangle$ with eigenvalues λ,

$$I_z|\lambda\rangle = \lambda|\lambda\rangle\ . \quad (23)$$

From the commutation relations (22) follows, that the states $I_\pm|\lambda\rangle$ are also eigenvectors of I_z,

$$I_z(I_\pm|\lambda\rangle) = (\lambda \pm 1)(I_\pm|\lambda\rangle). \quad (24)$$

For a finite dimension representation there must exist the largest eigenvalue, say j, such that

$$I_+|j\rangle = 0. \quad (25)$$

Similarly, there should also exist the smallest eigenvalue, such that

$$(I_-)^{(N+1)}|j\rangle = 0, \quad (26)$$

where N is some integer number.

Thus the eigenvalues of I_z is the sequence

$$j, j-1, j-2, ..., j-N. \quad (27)$$

The trace of the generator I_z is equal zero[2],

$$\begin{aligned}\text{trace}(I_z) &= j + (j-1) + ... + (j-N) \\ &= \frac{1}{2}(2j-N)(N+1) = 0\ . \end{aligned} \quad (28)$$

Thus $j = N/2$ and the eigenvalues of I_z are $j, j-1, ..., -j$. The dimension of a representation with a given j is $2j+1$.

Exercises

1. A group parameter φ is called additive if $g(\varphi_1)g(\varphi_2) = g(\varphi_1 + \varphi_2)$. Find the additive parameter of the Lorentz boosts along the z-axis. Find the corresponding generator.
2. Show that if $U = \exp(-i\sum_k J_k\alpha_k)$ is unitary ($U^\dagger U = 1$) and the parameters α_k are real, then the operators J_k are hermitian ($J_k^\dagger = J_k$). Show that if $\det(U) = 1$ then $\text{trace}(J_k) = 0$.
3. From the commutation relations

   $$[I_j, I_k] = i\sum_l \epsilon_{jkl}I_l$$

   find the 2x2 representation of the generators $\vec{I}$ (assuming I_3 is diagonal and I_1 is real).

[2] taking trace of the commutation relation