Non-relativistic limit of QFT

In the non-relativistic limit the QFT expression for the S-matrix reduces to the Lippmann-Schwinger equation for the scattering amplitude. The Lippmann-Schwinger equation is equivalent to the Schrödinger equation.

In the first order Born approximation the elastic scattering amplitude is given by the Fourier transform of the potential. Inversely, the potential is given as inverse Fourier transform of the scattering amplitude. The prescription to obtain the non-relativistic potential from a QFT is then relatively simple: calculate the relativistic transition amplitude for the elastic scattering in the lowest order perturbation theory; make the non-relativistic limit; and calculate the inverse Fourier transform.

Lippmann-Schwinger equation

Consider a non-relativistic elastic scattering in a system of two particles described by the Hamiltonian

$$H = H_0 + V \ , \ H_0 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial \mathbf{r}^2} \tag{1}$$

where $V(\mathbf{r})$ is the interaction potential, m is the reduced mass of the two particles, $\mathbf{r}$ is their relative distance, and $\hbar$ is the Planck's constant.

The cross-section of the scattering from the initial state with relative momentum $\mathbf{k}$ into the final state with relative momentum $\mathbf{k}'$ is determined by the so called reaction matrix[1] denoted here as M (for certain reasons which are of little importance in the present context),

$$d\sigma \propto \left|\langle \mathbf{k}'|M|\mathbf{k}\rangle\right|^2 . \tag{2}$$

It follows directly from the Schrödinger equation[2] that the reaction matrix satisfies the so called Lippmann-Schwinger equation,

$$M = V + VG_0M \ , \tag{12}$$

where G_0 is the free Green's function (with the correct boundary condition),

$$G_0 = \frac{1}{E - H_0 + i0} . \tag{13}$$

The solution of the Lippmann-Schwinger equation can be written as perturbation series in V,

$$M = V + VG_0V + VG_0VG_0V + \dots \ , \tag{14}$$

where the first term is the Born approximation,

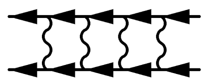
$$\langle \mathbf{k}'M\mathbf{k}\rangle = \langle \mathbf{k}'V\mathbf{k}\rangle = \int d^3rV(\mathbf{r})e^{i\mathbf{qr}} \ , \tag{15}$$

where $\mathbf{q} = \mathbf{k} - \mathbf{k}'$ is the transferred momentum.

It is actually the series (14) that the QFT expression for the reaction amplitude reduces to in the non-relativistic limit.

Ladder diagrams

One can show that only the so called ladder diagrams,



, (16)

survive in the non-relativistic limit.

[1] the reaction matrix is, of course, proportional to $S - 1$, where S is the S-matrix.

[2] In the non-relativistic quantum mechanics calculating the cross-section $d\sigma$ involves solving the the Schrödinger equation

$$(E - H_0)\psi^{(+)}_{\mathbf{k}} = V\psi^{(+)}_{\mathbf{k}} \tag{3}$$

where $\psi^{(+)}_{\mathbf{k}}$ is the scattering wave-function with the asymptotic

$$\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) \overset{r\to\infty}{\longrightarrow} e^{i\mathbf{kr}} + f\frac{e^{+ikr}}{r} . \tag{4}$$

where f is the scattering amplitude and the cross section is given as $d\sigma/d\Omega = |f|^2$.

Multiplying the Schrödinger equation by $(E - H_0)^{-1}$ from the left leads to the Lippmann-Schwinger equation for the scattering wave-function,

$$|\psi^{(+)}_{\mathbf{k}}\rangle = |\mathbf{k}\rangle + G^{(+)}_0(k)V|\psi^{(+)}_{\mathbf{k}}\rangle \ , \tag{5}$$

where $|\mathbf{k}\rangle$ is the free plane-wave solution with momentum $\mathbf{k}$,

$$H_0|\mathbf{k}\rangle = \frac{\hbar^2\mathbf{k}^2}{2m}|\mathbf{k}\rangle \tag{6}$$

and where G_0 is the free Green's function (with the correct boundary condition of outgoing waves),

$$G^{(+)}_0(k) = \frac{1}{E_k - H_0 + i0} \ , \ E_k = \frac{\hbar^2\mathbf{k}^2}{2m} . \tag{7}$$

Using the explicit expression for the free Green's function in coordinate space,

$$\langle \mathbf{r}'|G^{(+)}_0(k)|\mathbf{r}\rangle = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{8}$$

$$\overset{r>>r'}{\longrightarrow} \frac{1}{4\pi}\frac{e^{ikr}}{r}e^{-i\mathbf{k}'\mathbf{r}'} \ , \tag{9}$$

where $\mathbf{k}' = \frac{k\mathbf{r}}{r}$ one can easily show that

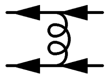
$$f \propto \langle \mathbf{k}'|V|\psi^{(+)}_{\mathbf{k}}\rangle . \tag{10}$$

The reaction matrix M is defined as the right-hand-side of (10)

$$\langle \mathbf{k}'|M|\mathbf{k}\rangle = \langle \mathbf{k}'|V|\psi^{(+)}_{\mathbf{k}}\rangle . \tag{11}$$

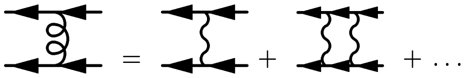
Multiplying (5) by $\langle \mathbf{k}'|V$ from the left and dropping the annoying superscripts immediately leads to (12).

The total scattering amplitude, denoted as



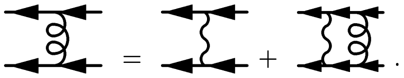
(17)

is apparently the sum of all ladder diagrams,



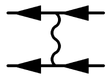
(18)

Equation (18) can be rewritten into the Lippmann-Schwinger equation,



(19)

Clearly (19) is equivalent to the Lippmann-Schwinger equation (12) if we assume

$V =$ 

(20)

In conclusion, in the non-relativistic limit a QFT reduces to the Schrödinger equation with the potential equal to the inverse Fourier transform of the lowest order elastic scattering amplitude.

One-boson exchange potential

The diagram (20) for a pseudo-scalar boson interaction Lagrangian $-g\bar{\psi}\gamma_5\psi\phi$ gives

$$M = i^2 g^2 \bar{u}(p_1')\gamma_5 u(p_1)\frac{1}{k^2-\mu^2}\bar{u}(p_2')\gamma_5 u(p_2) \quad (21)$$

The Dirac bispinor $u_{\mathbf{p}}$ is

$$u_{\mathbf{p}} = \begin{pmatrix} \phi_{\mathbf{p}} \\ \frac{\vec{\sigma}\mathbf{p}}{E_{\mathbf{p}}+m}\phi_{\mathbf{p}} \end{pmatrix} \quad (22)$$

which gives

$$\bar{u}(p_1')\gamma_5 u(p_1) = i\phi^{\dagger}_{\mathbf{p}_1'}\frac{\vec{\sigma}\mathbf{p}_1}{E_{\mathbf{p}_1}+m}\phi_{\mathbf{p}_1} \quad (23)$$

$$-i\phi^{\dagger}_{\mathbf{p}_1'}\frac{\vec{\sigma}\mathbf{p}_1'}{E_{\mathbf{p}_1'}+m}\phi_{\mathbf{p}_1} \quad (24)$$

In the non-relativistic limit $E \approx m$ up to the terms v^2

$$\bar{u}(p_1')\gamma_5 u(p_1) \approx \frac{i}{2m}\phi^{\dagger}_{\mathbf{p}_1'}\vec{\sigma}_1(\mathbf{p}_1-\mathbf{p}_1')\phi_{\mathbf{p}_1} \quad (25)$$

Now introducing $\mathbf{p}_1 - \mathbf{p}_1' = -\mathbf{q}$ gives

$$\bar{u}(p_1')\gamma_5 u(p_1) \approx -\frac{i}{2m}\phi^{\dagger}_{\mathbf{p}_1'}\vec{\sigma}_1\mathbf{q}\phi_{\mathbf{p}_1} \quad (26)$$

$$\bar{u}(p_2')\gamma_5 u(p_2) \approx \frac{i}{2m}\phi^{\dagger}_{\mathbf{p}_1'}\vec{\sigma}_2\mathbf{q}\phi_{\mathbf{p}_1} \quad (27)$$

In the c.m. frame

$$k^2 - \mu^2 = -(\mathbf{q}^2+\mu^2) \quad (28)$$

and finally

$$M = \frac{g^2}{(2m)^2}\phi_1^{\dagger}\phi_2^{\dagger}\frac{(\vec{\sigma}_1\vec{q})(\vec{\sigma}_2\vec{q})}{\vec{q}^2+\mu^2}\phi_1\phi_2 \quad (29)$$

The one-pseudo-scalar-boson-exchange-potential is the given as

$$V_{\mathrm{PS}}(\mathbf{r}) = -\frac{g^2}{2m^2}\sigma_{1a}\sigma_{2b}\int\frac{d^3r}{(2\pi)^3}\frac{q_a q_b}{\mathbf{q}^2+\mu^2}e^{i\mathbf{qr}} \quad (30)$$

The integral

$$\int\frac{d^3r}{(2\pi)^3}\frac{q_a q_b}{\mathbf{q}^2+\mu^2}e^{i\mathbf{qr}} = \quad (31)$$

$$-\frac{\partial}{\partial r_a}\frac{\partial}{\partial r_b}\int\frac{d^3r}{(2\pi)^3}\frac{1}{\mathbf{q}^2+\mu^2}e^{i\mathbf{qr}} = \quad (32)$$

$$-\frac{\partial}{\partial r_a}\frac{\partial}{\partial r_b}\frac{1}{4\pi}\frac{e^{-\mu r}}{r} = \quad (33)$$

$$\frac{\mu^2}{4\pi}\frac{e^{-\mu r}}{r}\left(\frac{1}{\mu r}+\frac{1}{(\mu r)^2}\right)\delta_{ab} \quad (34)$$

$$-\frac{\mu^2}{4\pi}\frac{e^{-\mu r}}{r}\left(1+\frac{3}{\mu r}+\frac{3}{(\mu r)^2}\right)\frac{r_a r_b}{r^2} \quad (35)$$

The OBEP with pseudo-scalar boson is thus a finite-range spin-spin and tensor potential of Yukawa type with the range equal to inverse mass of the exchange boson.

The OBEP with a vector boson has a slightly different spin structure which in addition includes central and spin-orbit forces. The central force has the Yukawa form $e^{-\mu r}/r$. In the limit of massless vector boson this gives the Coulomb central potential.