Examples of Feynman diagrams

For the interaction Lagrangian $\mathcal{L}_v = -g\bar{\psi}\psi\phi$ the second order term of the S-matrix is given as

$$S^{(2)} = \frac{(ig)^2}{2!}\int d^4x_1 d^4x_2 \mathcal{T}\bar{\psi}(x_1)\psi(x_1)\phi(x_1)\cdot \bar{\psi}(x_2)\psi(x_2)\phi(x_2)\ , \qquad (1)$$

where $\mathcal{T}$ denotes T-product. The Wick's expansion of the T-product in (1) gives (among others) the term

$$\mathcal{N}\bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2)\underline{\phi(x_1)\phi(x_2)}\ , \qquad (2)$$

where $\mathcal{N}$ denotes normal ordering of operators. According to the Feynman rules the corresponding diagram, see figure 1, has two vertexes, x_1 and x_2, connected by a bosonic contraction $\underline{\phi(x_1)\phi(x_2)}$, and non-contracted fermionic lines $\bar{\psi}(x_1)$, $\psi(x_1)$ and $\bar{\psi}(x_2)$, $\psi(x_2)$ attached correspondingly to vertexes x_1 and x_2.

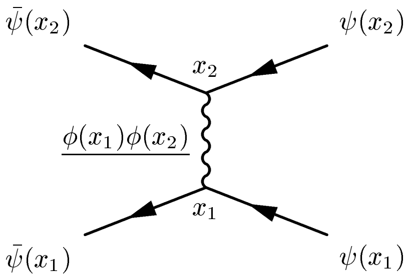


Figure 1: The Feynman diagram representing the term (2).

Another interesting term in the Wick's expansion of the T-product in (1) is

$$\mathcal{N}\bar{\psi}(x_1)\underline{\psi(x_1)\bar{\psi}(x_2)}\psi(x_2)\phi(x_1)\phi(x_2)\ . \qquad (3)$$

Its Feynman diagram, see figure 2, has two vertexes, connected by the fermionic contraction $\underline{\psi(x_1)\bar{\psi}(x_2)}$ and the non-contracted fields $\bar{\psi}(x_1)$, $\phi(x_1)$ (with vertex x_1) and $\psi(x_2)$, $\phi(x_2)$ (with vertex x_2).

Initial and final states for Feynman diagrams

An non-contracted field $\mathcal{N}\psi$ gives a non-zero result when acting on a particle state $a^\dagger_{\mathbf{p}\lambda}|0\rangle$ to the right,

$$\mathcal{N}\psi(x)a^\dagger_{\mathbf{p}\lambda}|0\rangle = u_{\mathbf{p}\lambda}e^{-ikx}|0\rangle \qquad (4)$$

or on the anti-particle $b^\dagger_{\mathbf{k}\lambda}|0\rangle$ state to the left,

$$\langle 0|b_{\mathbf{p}\lambda}\mathcal{N}\phi(x) = \langle 0|v_{\mathbf{p}\lambda}e^{+ikx} \qquad (5)$$

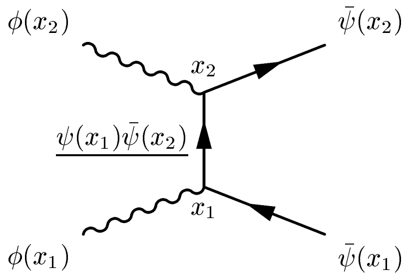


Figure 2: The Feynman diagram representing the term (3).

An non-contracted field $\mathcal{N}\bar{\psi}$ acts non-vanishingly on an anti-particle to the right,

$$\mathcal{N}\bar{\psi}(x)b^\dagger_{\mathbf{p}\lambda}|0\rangle = \bar{v}_{\mathbf{p}\lambda}e^{-ipx}|0\rangle \qquad (6)$$

or a particle to the left,

$$\langle 0|a_{\mathbf{p}\lambda}\mathcal{N}\bar{\psi}(x) = \langle 0|\bar{u}_{\mathbf{p}\lambda}e^{+ipx} \qquad (7)$$

Thus an non-contracted field $\mathcal{N}\psi$ can represent a particle in the initial state or an antiparticle in the final state. The barred field $\mathcal{N}\bar{\psi}$ can represent an antiparticle in the initial state or a particle in the final state.

An non-contracted real field $\mathcal{N}\phi$ can represent a particle in both the initial and or final states.

If the initial state is assumed to be at the right side of the diagram and the final state at the left side, then figure 1 describes (a contribution to) fermion-fermion scattering, figure 2 – fermion-fermion annihilation into two bosons.

If the initial state is assumed at the bottom of the diagram and the final state at the top, then figure 1 describes (a contribution to) fermion-antifermion scattering, figure 2 – boson-fermion scattering.

CPT theorem

The CPT theorem states that in the canonical quantum field theory the action is invariant under the combination of charge conjugation, parity and time reversal. The consequence is that the cross sections of certain reactions should then be related.

Let us see how it works with the scalar field.

C: charge conjugation

Under charge conjugation the particles are exchanged with antiparticles,

$$a_{\mathbf{k}} \xrightarrow{C} b_{\mathbf{k}}\ . \qquad (8)$$

The field $\phi(x)$ then turns into $\phi^\dagger(x)$,

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ikx} + b^\dagger_{\mathbf{k}} e^{+ikx} \right) \stackrel{C}{\to} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(b_{\mathbf{k}} e^{-ikx} + a^\dagger_{\mathbf{k}} e^{+ikx} \right) = \phi^\dagger(x) \ . \tag{9}$$

P: parity transformation

Under parity transformation the momentum changes sign,

$$a_{\mathbf{k}} \stackrel{P}{\to} a_{-\mathbf{k}} \ . \tag{10}$$

The field $\phi(t, \mathbf{x})$ then turns into $\phi(t, -\mathbf{x})$,

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ikx} + b^\dagger_{\mathbf{k}} e^{+ikx} \right) \stackrel{P}{\to} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{-\mathbf{k}} e^{-ikx} + b^\dagger_{-\mathbf{k}} e^{+ikx} \right) = \phi(t, -\mathbf{x}) \ . \tag{11}$$

T: time reversal

Under time reversal the process of an annihilation of a particle turns into a process of generation of a particle with opposite momentum,

$$a_{\mathbf{k}} \stackrel{T}{\to} a^\dagger_{-\mathbf{k}} \ . \tag{12}$$

The field $\phi(t, \mathbf{x})$ then turns into $\phi^\dagger(-t, \mathbf{x})$,

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a_{\mathbf{k}} e^{-ikx} + b^\dagger_{\mathbf{k}} e^{+ikx} \right) \stackrel{T}{\to} \sum_{\mathbf{k}} \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left(a^\dagger_{-\mathbf{k}} e^{-ikx} + b_{-\mathbf{k}} e^{+ikx} \right) = \phi^\dagger(-t, \mathbf{x}) \ . \tag{13}$$

CPT

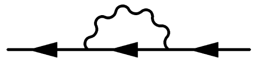
Under the combination of all three the field ϕ changes the sign of the argument,

$$\phi(x) \stackrel{CPT}{\longrightarrow} \phi(-x) \ . \tag{14}$$

However, since the action involves integration $\int d^4x$ the sign change of the argument apparently leaves it unchanged, q.e.d.

Exercises

1. Formulate Feynman rules for $\mathcal{L}_v = -g\bar{\psi}\psi\phi$ in momentum space:
   (a) Consider e.g. "Compton scattering" process, diagram Figure 2 between certain plane-waves, and carry out coordinate integration.
   (b) Figure out the correspondence between the terms in the S-matrix element and the elements of the diagram.
   (c) Consider the loop diagram



   and write down the corresponding matrix element in momentum space. Figure out the expression for the loop. Also estimate the asymptotic behavior of the loop integral.