Examination problem

Consider a standard Yang-Mills theory with the Lagrangian[1]

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi - \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} \; . \tag{1}$$

1. Write down the Euler-Lagrange equations for fermions and gauge bosons.
2. Find the conserved currents of the theory.

Minimal solution

Fermionic Euler-Lagrange equation

The canonical Euler-Lagrange equation for ψ,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = \frac{\partial \mathcal{L}}{\partial \bar{\psi}} \; , \tag{2}$$

immediately gives

$$\underline{\underline{(i\gamma^\mu D_\mu - m)\psi = 0 \; .}} \tag{3}$$

Bosonic Euler-Lagrange equation

The canonical Euler-Lagrange equation for A^e_μ is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^e_\nu)} = \frac{\partial \mathcal{L}}{\partial A^e_\nu} \; . \tag{4}$$

The derivatives of the Lagrangian with respect to the fields can be calculated as

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^e_\nu)} = -\frac{1}{2}F^{a\mu'\nu'}\frac{\partial F^a_{\mu'\nu'}}{\partial(\partial_\mu A^e_\nu)} = -F^{e\mu\nu} \; , \tag{5}$$

$$\frac{\partial \mathcal{L}}{\partial A^e_\nu} = -g\bar{\psi}\gamma^\nu T^e \psi - \frac{1}{2}F^{a\mu'\nu'}\frac{\partial F^a_{\mu'\nu'}}{\partial A^e_\nu} = \\ -g\bar{\psi}\gamma^\nu T^e\psi - F^{a\mu\nu}(-g f_{abe}A^b_\mu) \; . \tag{6}$$

Which leads to

$$\underline{\underline{-\partial_\mu F^{e\mu\nu} = -g j^{e\nu} + g f_{abe}A^b_\mu F^{a\mu\nu} \; ,}} \tag{7}$$

where $j^{e\nu} = \bar{\psi}\gamma^\nu T^e \psi$.

When the structure constants f_{abc} are zero, equation (7) reduces to the ordinary Maxwell equation.

[1] where $D_\mu = \partial_\mu + igA_\mu$, $A_\mu = A^a_\mu T^a$, $[T^a, T^b] = if_{abc}T^c$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - g f_{abc}A^b_\mu A^c_\nu T^a$

Conserved currents

Multiplying euation (7) from the left with ∂_ν gives[2] the conserved currents,

$$\underline{\underline{\partial_\nu(j^{e\nu} - f_{abe}A^b_\mu F^{a\mu\nu}) = 0 \; .}} \tag{8}$$

[2] $\partial_\mu\partial_\nu F^{\mu\nu} = 0$