Group Theory in Quantum Mechanics
Exam Problems

Jeppe Dakin
Department of Physics and Astronomy, Aarhus University, Denmark
dakin@phys.au.dk

March 29, 2017

1

There are five point symmetry groups in three dimensions with order 6. In Schönflies notation, they are denoted C_6, $\mathrm{C}_{3\mathrm{h}}$, $\mathrm{C}_{3\mathrm{v}}$, D_3 and S_6.

(a) Counting the identity as a rotation, how many rotations do each group have? How many reflections? How many rotoreflections?

We will examine each of the five groups in turn.

C_6

This is the cyclic group of order 6. In 2D, this is the symmetry group of a regular hexagon with directed sides. Clearly this group contains the six rotations around the center of the hexagon, call them c^i, $i = 1, 2, \ldots, 6$. Taking $c \equiv c^1$ to be the smallest non-zero rotation, this must be a rotation of $2\pi/6$. A general rotation by $2\pi/6 \times i$ is then indeed c^i, where the superscript can now be interpreted as a power. In particular, $c^6 = e$. Thus a full rotation of 2π is the same as doing nothing.

Since the hexagon have directed sides, no mirror planes perpendicular to the plane of the hexagon exist. Because we consider the hexagon to reside in 2D space, the plane of the hexagon must not be considered a mirror plane. We conclude that this group consists purely of the rotations, and hence its presentation is

$$\mathrm{C}_6 = \left\{\, c \;\middle|\; c^6 = e \,\right\}.$$

The group is thus generated by a single element c, which is exactly what is meant by a *cyclic* group.

To summarize, the group contain only rotations and thus no reflections and no rotoreflections. We may write this summary as

$$\mathrm{C}_6 = \begin{cases} \text{rotations :} & e,\ c,\ c^2,\ c^3,\ c^4,\ c^5 \qquad (6) \\ \text{reflections :} & \qquad\qquad\qquad\qquad\quad (0) \\ \text{rotoreflections :} & \qquad\qquad\qquad\qquad\quad (0) \end{cases}$$

$\mathrm{C}_{3\mathrm{h}}$

The name of this group implies that it is constructed by appending a "*h*orizontal" reflection, call it σ_h, to the group C_3, where C_3 is analogous to C_6, but replacing the regular hexagon with a regular (equilateral) triangle, still remembering to make the sides directed. To append the reflection σ_h, we now embed the triangle in 3D space, giving the triangle an upper and a lower side. If these two sides are similar, reflection in the plane of the triangle itself is a new symmetry, which is the one we associate with σ_h. Now that we are dealing with 3D space, we might as well make the geometrical object 3D as well. This can be done by simply extruding the triangle along the newly added dimension. Thus the set of elements $\{c,\, c^2,\, c^3 = e,\, \sigma_\mathrm{h}\}$, where c is now a rotation by $2\pi/3$, is a subset of $\mathrm{C}_{3\mathrm{h}}$. As this is supposed to be a group of order 6, we miss 2 elements. That is, our set of 4 elements is not closed under composition, and thus do not form a group. The remaining 2 elements can then be constructed from c and σ_h. Geometrically it is clear that doing a reflections *and* a rotation results in a new symmetry transformation, and so we may take the remaining elements as $c\sigma_\mathrm{h}$ and $c^2\sigma_\mathrm{h}$. As these are compositions of rotations and reflections, they are rotoreflections. The group elements can then be categorized as

$$\mathrm{C}_{3\mathrm{h}} = \begin{cases} \text{rotations :} & e,\, c,\, c^2 \qquad (3) \\ \text{reflections :} & \sigma_\mathrm{h} \qquad (1) \\ \text{rotoreflections :} & c\sigma_\mathrm{h},\, c^2\sigma_\mathrm{h} \qquad (2) \end{cases}$$

$\mathrm{C}_{3\mathrm{v}}$

This group is similar to $\mathrm{C}_{3\mathrm{h}}$, but now the reflection σ_v is "*v*ertical", meaning perpendicular to the plane of the triangle. In 2D space, this corresponds to the symmetry group of an equilateral triangle, this time with *un*directed sides. Disregarding the third dimension eliminate the horizontal reflection, and disregarding the directiveness of the sides allow for vertical mirror planes. We can bring the triangle into 3D space by extruding once gain, but this time we have to make these vertical faces directed *vertically*, in order not to create an additional horizontal reflection symmetry. In analogy with $\mathrm{C}_{3\mathrm{h}}$, the elements of $\mathrm{C}_{3\mathrm{v}}$ are then $\{e,\, c,\, c^2,\, \sigma_\mathrm{v},\, c\sigma_\mathrm{v},\, c^2\sigma_\mathrm{v}\}$. In contrast to $\mathrm{C}_{3\mathrm{h}}$, the elements $c\sigma_\mathrm{v}$ and $c^2\sigma_\mathrm{v}$ are not rotoreflections, but just reflections. This is clear geometrically, as the extruded triangle have not just one vertical mirror plane (corresponding to σ_v), but actually three vertical mirror planes. These two additional mirror symmetries can only be associated with $c\sigma_\mathrm{v}$ and $c^2\sigma_\mathrm{v}$, which makes sense as the mirror planes are situated at an angle of $2\pi/3$ relative to each other. Thus

$$\mathrm{C}_{3\mathrm{v}} = \begin{cases} \text{rotations :} & e,\, c,\, c^2 \qquad (3) \\ \text{reflections :} & \sigma_\mathrm{v},\, c\sigma_\mathrm{v},\, c^2\sigma_\mathrm{v} \qquad (3) \\ \text{rotoreflections :} & \qquad (0) \end{cases}$$

D_3

To construct D_3, we once again start with C_3, which in 2D can be represented as the symmetries of an equilateral triangle with directed sides. The D stands for *d*ihedral, meaning that we should remove the restriction of directed sides. The added symmetries are then three reflections; those of $\mathrm{C}_{3\mathrm{v}}$. If we now embed the triangle in 3D space, the possibility of rotating the triangle by π through three axes in its plane arise, creating three additional symmetries. We now wish to eliminate the vertical reflections, which can be done e.g. by coloring the vertical faces in such a manner as to break the reflection symmetry but retaining the rotational symmetry about axes perpendicular to these faces.

Denote one of the new π-rotations by b, and the remaining two rotations can be taken to be cb and c^2b. It should be noted that although generally physically distinct, the transformation b on the extruded triangle is equivalent to reflection in both the vertical and horizontal plane, $b = \sigma_\mathrm{h}\sigma_\mathrm{v}$. In total,

$$\mathrm{D}_3 = \begin{cases} \text{rotations :} & e,\ c,\ c^2,\ b,\ cb,\ c^2b \quad (6) \\ \text{reflections :} & \quad (0) \\ \text{rotoreflections :} & \quad (0) \end{cases}$$

S_6

As with C_6, we may here think of an extruded, regular hexagon. This have horizontal reflection symmetry, which we wish to reduce to rotoreflection symmetry. We can achieve this by coloring each vertical face the horizontal mirror reversed of its neighbouring face(s). This also reduces the C_6 symmetry down to C_3. The group elements in S_6 are then the three rotations e, c^2, c^4, where c is a 2π/6 rotation, together with the three rotoreflections $c\sigma_\mathrm{h}$, $c^3\sigma_\mathrm{h}$, $c^5\sigma_\mathrm{h}$.

Note that since opposing vertical faces are now painted differently, it does not matter whether the sides of the original hexagon were directed or not.

In total, we have

$$\mathrm{S}_6 = \begin{cases} \text{rotations :} & e,\ c^2,\ c^4 \quad (3) \\ \text{reflections :} & \quad (0) \\ \text{rotoreflections :} & c\sigma_\mathrm{h},\ c^3\sigma_\mathrm{h},\ c^5\sigma_\mathrm{h} \quad (3) \end{cases}$$

(b) For each of the five groups, draw or find a picture of an object with that point symmetry group.

For each of the five point groups is drawn a 3D objects in 3D space with the corresponding symmetry group. The chosen object is the triangular or hexagonal prism described in (a) for each group. Directiveness of the sides are drawn as arrows on the vertical faces. Instead of coloring in part of the faces, these arrows are simply placed asymmetrically.

For C_6, a directed, regular hexagon in 2D was described, as we never needed the third dimension. We now bring this object into 3D space and extrude it, making a hexagonal prism. The directiveness of the hexagon is now drawn as an arrow on the extruded sides. To remove any additional symmetries created by the 2D → 3D transformation, we do not place the arrow symmetrically in the center.

The 3D objects for the remaining four groups are drawn in a similar fashion. They can all be seen in figure 1.

(c) Which of these groups (if any) are abelian?

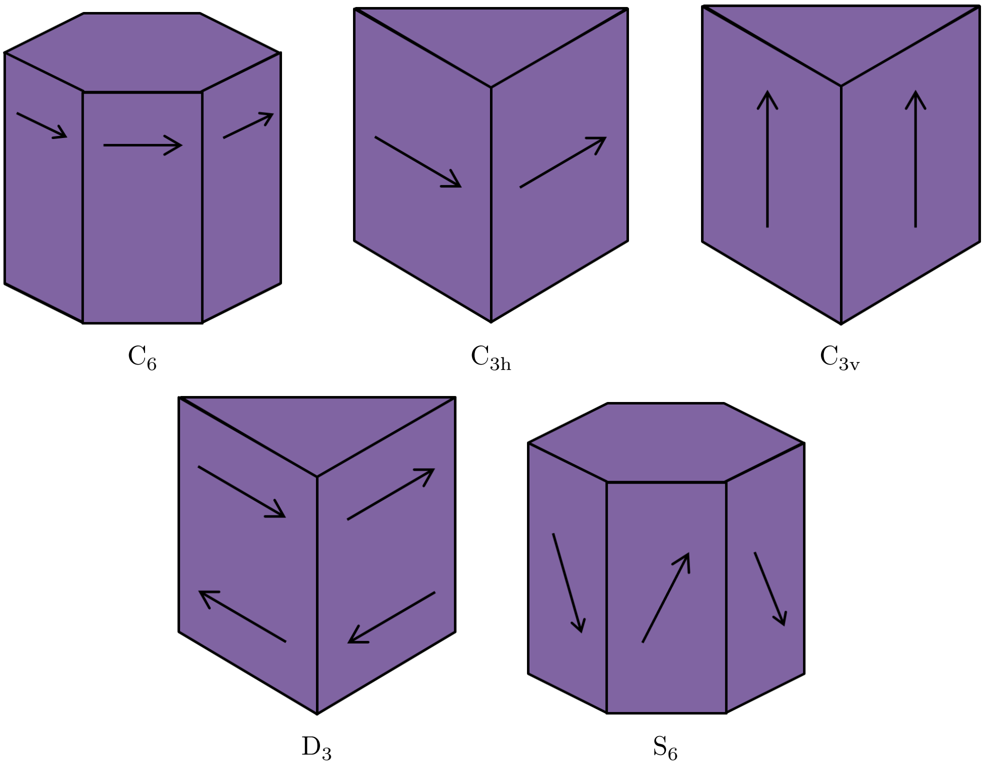


Figure 1 – 3D objects with point symmetry groups.

As C_6 is cyclic and thus generated from a single element c, it is clearly abelian.

Even though the group C_{3h} includes horizontal reflection σ_h as well as rotations c (and combinations thereof — rotoreflections), these transformations commute and so C_{3h} is abelian as well. The commutativity of c and σ_h is a consequence of the fact that the mirror plane of σ_h is perpendicular to the axis of rotation of c. That is, σ_h only operates along the vertical axis, precisely the one which is left invariant by c.

The C_{3v} group is non-abelian, as the reflections σ_v are not perpendicular to the axis of rotation of c. Thus generally, a reflection followed by a rotation is not the same as a rotation followed by a reflection.

The D_3 group consists solely of rotations, but through four distinct axes. As rotations through different axes do not commute, D_3 is non-abelian.

The S_6 consists of rotations as well as rotoreflections. The rotational part of the rotoreflections is along the same axis as the pure rotations, and the reflective part is through a mirror plane perpendicular to the axis of rotation. Thus the rotoreflections commute with the rotations, implying that S_6 is abelian.

In summary:

$$\text{abelian?}\begin{cases} \mathrm{C_6} & \checkmark \\ \mathrm{C_{3h}} & \checkmark \\ \mathrm{C_{3v}} & \times \\ \mathrm{D_3} & \times \\ \mathrm{S_6} & \checkmark \end{cases}$$

(d) Which of these groups (if any) are isomorphic to each other?

Our goal here is to find all abstract groups of order 6, and then determine which of these abstract groups each of the five point groups are isomophic to.

From Cayley's theorem, we know that a group of order $n < \inf$ is isomorphic to a subgroup of S_n, where S_n is the symmetric group, the group of permutations on n distinct symbols. Setting $n = 6$, we get S_6, the group of permutations on 6 distinct symbols, which is not to be confused with our point group with the same name. We thus search for abstract subgroups $H \subset S_6$ with order 6, $[H] = 6$. Remembering that the order of each element (permutation) in H has to be a divisor of $n = 6$, we have $[h] \in \{1, 2, 3, 6\}\,\forall h \in H$, where $[h]$ is the order of the element h. We should then build up H from transpositions (2-cycles), 3-cycles and 6-cycles. Note that the trivial 1-cycle is the identity element, which can here be thought of as a trivial permutation, in cycle-notation written (). As this will be an element of any H we might find, we do not have to think about it.

The 6-cycle is of course unique up to the labelling of the symbols:

$$\begin{aligned} \mathrm{S}_6 \ni \begin{array}{ccc} & 3 \leftarrow 2 & \\ 4 & & 1 \\ & 5 \rightarrow 6 & \end{array} &= (1\,2\,3\,4\,5\,6) \\ &= c_6\,, \end{aligned} \tag{1.1}$$

where the actual cycle is shown in order to define the cycle notation to the right. The name c_6 is given, meaning a cycle of order 6, which as already stated is unique in S_6. Repeated application of c_6 corresponds to cycling around in the cycle (1.1), and so $c_6^6 = () = e$. It is clear then that c_6 generates a cyclic group of order 6, and that this is the abstract group of which C_6 is isomorphic. The name of this abstract group is Z_6.

The next possibility for $[h]$ is 3, resulting a 3-cycle $(1\,2\,3) = c_3$. We are thus in need of three more elements to form a group of order 6. We therefore add in a new element b. We cannot have $[b] = 1$ since this is the identity $e = c_3^3$, and so in that case b would not be a *new* element. Also, $[b]$ cannot be 6 as this would make $[H] > 6$. Thus b is of order 2 or 3. If $[b] = 2$, we have the 6 elements $\{e,\, c_3,\, c_3^2,\, b,\, bc_3,\, bc_3^2\}$. Thus if $[b] = 3$, b^2 must be one of the other element if we wish to keep the group order at 6. We can check explicitly that this does not work out:

$$[b] = 3 \Rightarrow b^2 = \begin{cases} e & \Rightarrow [b] = 2\,, \\ b & \Rightarrow b = e\,, \\ bc_6 & \Rightarrow b = c_3\,, \\ bc_6^2 & \Rightarrow b = c_3^2\,, \\ c_6 & \Rightarrow bc_3 = e\,, \\ c_6^2 & \Rightarrow bc_3^2 = e\,, \end{cases}$$

and so we must have $[b] = 2$. We could now go on and show that the set $\{e,\, c_3,\, c_3^2,\, b,\, bc_3,\, bc_3^2\}$ with $c_3^3 = e = b^2$ is closed under compositions, i.e. that elements such as $c_3 b$ are already contained in the set. We know however from problem (c) that non-abelian groups of order 6 exists, and so this has to be the one, as this is the only candidate. We thus will not bother constructing the Cayley table, but let us for completeness show that this group is non-abelian. We do this by assuming that c_3 and b commute:

$$c_3 b = bc_3 \Rightarrow \begin{cases} (bc_3)^2 &= c_3^2\,, \\ (bc_3)^3 &= b\,, \\ (bc_3)^4 &= c_3\,, \\ (bc_3)^5 &= bc_3^2\,, \\ (bc_3)^6 &= e\,, \end{cases}$$
$$\Rightarrow [bc_3] = 6\,,$$

which is against the assumption of group elements being of order 1, 2 and 3.

With only two abstract groups of order 6, one abelian and one not, we can easily conclude that all the abelian point groups are isomorphic to each other, and the non-abelian are isomorphic to each other:

$$\mathrm{C}_6 \cong \mathrm{C}_{3\mathrm{h}} \cong \mathrm{S}_6\,,$$
$$\mathrm{C}_{3\mathrm{v}} \cong \mathrm{D}_3\,.$$

(e) Which of these groups (if any) are simple, i.e. which have no normal subgroups?

Generally, a subgroup $H \subset G$ is normal if it closed under conjugation by elements of G, $gHg^{-1} = H\,\forall g \in G$. If G is abelian, H inherits this property and thus $gHg^{-1} = H$ is given for every subgroup H.

The abelian C_6 group have proper subgroups C_2 and C_3, generated by c^3 and c_2, respectively. Thus both C_2 and C_3 are normal subgroups of C_6.

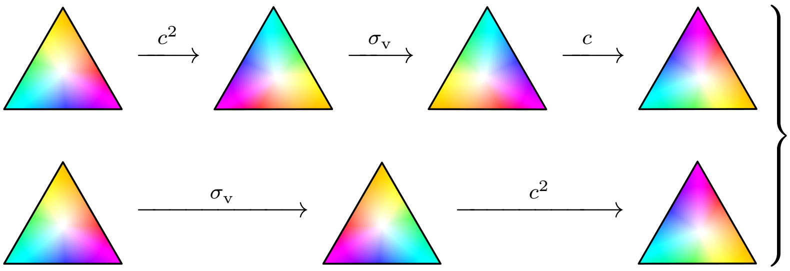
The abelian $\mathrm{C}_{3\mathrm{h}}$ group also have the proper subgroup C_3, which is then normal. Another subgroup of $\mathrm{C}_{3\mathrm{h}}$ is $\mathrm{C}_\mathrm{s} \cong \mathrm{Z}_2$, generated by σ_h. The rotoreflections $c\sigma_\mathrm{h}$ and $c^2\sigma_\mathrm{h}$ do not participate in further subgroups, as these actually generate the entirety of $\mathrm{C}_{3\mathrm{h}}$.

The last abelian group is S_6, which have the proper subgroups C_3, generated by both c^2 and c^4 individually, where c is now a rotation by $2\pi/6$, not $2\pi/3$. Because 6 is even, S_6 contains the rotoreflection $c^3\sigma_\mathrm{h}$ which rotates through π. This transformation corresponds to inversion, which generate its own subgroup $\mathrm{C}_\mathrm{i} \cong \mathrm{Z}_2$, which is then also normal.

The non-abelian group $\mathrm{C}_{3\mathrm{v}}$ also contains both $\mathrm{C}_3 = \{e,\, c,\, c^2\}$ and $\mathrm{C}_\mathrm{s} = \{e,\, \sigma_\mathrm{v}\} \cong \{e,\, c\sigma_\mathrm{v}\} \cong \{e,\, c^2\sigma_\mathrm{v}\}$ as subgroups. However, since the rotations do not commute with the reflections (the axis of rotation is not perpendicular to the mirror planes), these subgroups might not be normal. It is however obvious that conjugating a rotation by a reflection is again some rotation. Since C_3 contains all the rotations of $\mathrm{C}_{3\mathrm{v}}$, we can conclude that C_3 is normal. Similarly, conjugating the reflection by a rotation in $\mathrm{C}_{3\mathrm{v}}$ will always result in some reflection. However, this reflection will not be the same as the one we started with, e.g. $c\sigma_\mathrm{v}c^{-1} = c\sigma_\mathrm{v}c^2 = c^2\sigma_\mathrm{v}$, where the last equality requires some mental effort. To calculate this, we could write out the V-representation of these transformations which acts on 3-vectors and simply carry out the matrix multiplication. However, let us do it geometrically/graphically. Since $\sigma_\mathrm{h} \notin \mathrm{C}_{3\mathrm{v}}$, all elements can be faithfully represented as transformations on a 2D polygon, specifically the triangular* prism of $\mathrm{C}_{3\mathrm{v}}$ from figure 1

*We thus have an isomorphism between $\mathrm{C}_{3\mathrm{v}}$ and the set of such triangles. The effect of the coloring is to make each vertex distinct. There then exists one colored triangle for each of the $3! = 6$ permutations of the three vertices.

viewed along the vertical axis. Coloring this triangle so that we may see the effects of the transformations, we have



$$\Rightarrow c\sigma_\mathrm{v}c^2 = c^2\sigma_\mathrm{v}\,.$$

where c is taken to be anti-clockwise and σ_v a left-right reflection. Thus conjugating reflections by rotations mixes the three subgroups $\{e,\,\sigma_\mathrm{v}\} \cong \{e,\,c\sigma_\mathrm{v}\} \cong \{e,\,c^2\sigma_\mathrm{v}\}$. I am unsure whether these subgroups then count as normal. It depend on whether we count each of these subgroups individually, or whether they count as one since they are all isomorphic.

The non-abelian group D_3 contain the proper subgroup C_3 generated by c as well as C_2 generated by b. Again, since c and b do not commute, these subgroups may not be normal. However, since the rotation axis for c and c^2 is vertical while all three axes of the rotations b, cb and c^2b lie in the horizontal plane, the rotations in C_3 are perpendicular to those in C_2. At the same time, all rotations in C_2 are two-fold rotations, and so applying two of them will cancel out in the sense that the overall rotation will not be in the horizontal plane. Thus both C_3 and C_2 are closed under conjugation by D_3, and so they are normal subgroups.

In summary:

$$\text{simple?}\begin{cases} \mathrm{C}_6 & ✗ \\ \mathrm{C}_{3\mathrm{h}} & ✗ \\ \mathrm{C}_{3\mathrm{v}} & (✓) \\ \mathrm{D}_3 & ✗ \\ \mathrm{S}_6 & ✗ \end{cases}$$

2

Consider the isotropic two-dimensional harmonic oscillator Hamiltonian

$$\begin{aligned} \hat{H} &= \frac{\hbar^2}{2m}\big(\hat{P}_1^2 + \hat{P}_2^2\big) + \frac{1}{2}m\omega^2\big(\hat{X}_1^2 + \hat{X}_2^2\big) \\ &= \hbar\omega\big(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1\big) \end{aligned}$$

where $\hat{X}_i$ are the position operators, $\hat{P}_i$ are the momentum operators and $\hat{a}_i^\dagger$ and $\hat{a}_i$ are the standard creation and annihilation operators.

(a) What is the spectrum of energies and degeneracies for this Hamiltonian?

Defining the number operators $\hat{N}_i = \hat{a}_i^\dagger\hat{a}_i$, the Hamiltonian can be written as

$$\hat{H} = \hbar\omega\big(\hat{N}_1 + \hat{N}_2 + 1\big)\,.$$

The creation and annihilation operators satisfy the standard (bosonic) commutation relations

$$\left[\hat{a}_i,\, \hat{a}_j^\dagger\right] = \delta_{ij}\,, \tag{2.1}$$

which translates into commutation relations between the number operators and the creation and annihilation operators:

$$\begin{aligned}
\left[\hat{N}_i,\, \hat{a}_j\right] &= \left[\hat{a}_i^\dagger \hat{a}_i,\, \hat{a}_j\right] \\
&= \hat{a}_i^\dagger \overbrace{\left[\hat{a}_i,\, \hat{a}_j\right]}^{0} + \overbrace{\left[\hat{a}_i^\dagger,\, \hat{a}_j\right]}^{-\delta_{ij}} \hat{a}_i \\
&= -\hat{a}_i \delta_{ij}\,, \qquad (2.2)\\
\left[\hat{N}_i,\, \hat{a}_j^\dagger\right] &= \left[\hat{a}_i^\dagger \hat{a}_i,\, \hat{a}_j^\dagger\right] \\
&= \hat{a}_i^\dagger \overbrace{\left[\hat{a}_i,\, \hat{a}_j^\dagger\right]}^{\delta_{ij}} + \overbrace{\left[\hat{a}_i^\dagger,\, \hat{a}_j^\dagger\right]}^{0} \hat{a}_i \\
&= \hat{a}_i^\dagger \delta_{ij}\,. \qquad (2.3)
\end{aligned}$$

We see that the set of operators $\left\{\hat{a}_i^\dagger,\, \hat{a}_i,\, \hat{N}_i,\, \delta_{ij}\right\}$ form a closed algebra.

Clearly $\hat{H}$ and $\hat{N}_1 + \hat{N}_2$ share the same eigenspace, and so we now wish to find the eigenstates of $\hat{N}_i$. Denote a general state by $|n_1,\, n_2\rangle$, then $\hat{N}_i\, |n_1,\, n_2\rangle = n_i\, |n_1,\, n_2\rangle$, for $i \in \{1,\, 2\}$. Now consider the action of $\hat{N}_i$ on the state $\hat{a}_i^\dagger\, |n_1,\, n_2\rangle$:

$$\begin{aligned}
\hat{N}_i\big(\hat{a}_i^\dagger\, |n_1,\, n_2\rangle\big) &= \big(\hat{a}_i^\dagger \hat{N}_i + \hat{a}_i^\dagger\big)\, |n_1,\, n_2\rangle \\
&= \hat{a}_i^\dagger \big(\hat{N}_i + 1\big)\, |n_1,\, n_2\rangle \\
&= \hat{a}_i^\dagger (n_i + 1)\, |n_1,\, n_2\rangle \\
&= (n_i + 1)\big(\hat{a}_i^\dagger\, |n_1,\, n_2\rangle\big)\,.
\end{aligned}$$

If $\hat{N}_i$ counts the number of quanta in the i'th dimension, $\hat{a}_i$ creates another such quanta. Similarly $\hat{a}_i$ can be found to annihilate a quanta in the i'th dimension:

$$\begin{aligned}
\hat{N}_i\big(\hat{a}_i\, |n_1,\, n_2\rangle\big) &= \big(\hat{a}_i \hat{N}_i - \hat{a}_i\big)\, |n_1,\, n_2\rangle \\
&= \hat{a}_i \big(\hat{N}_i - 1\big)\, |n_1,\, n_2\rangle \\
&= \hat{a}_i (n_i - 1)\, |n_1,\, n_2\rangle \\
&= (n_i - 1)\big(\hat{a}_i\, |n_1,\, n_2\rangle\big)\,.
\end{aligned}$$

Now then, given a single state with non-zero n_i, the creation and annihilation operators can generate the full spectrum. We postulate that a "vacuum" state $|0,\, 0\rangle$ exists such that the states (and hence the energies) are bounded from below. This also implies that the eigenvalues n_i are whole numbers. The spectrum is then

$$E_{n_1, n_2} = \hbar\omega(n_1 + n_2 + 1)\,, \quad n_1,\, n_2 \in \mathbb{N}\,.$$

If we define $n \equiv n_1 + n_2$, then for each energy E_n, we can choose n_1 to be any of the $n+1$ integers $0 \leq n_1 \leq n$, after which n_2 can only be chosen as $n_2 = n - n_1$. The degeneracies for the Hamiltonian is thus $g(n) = n + 1$.

(b) Show that this Hamiltonian has O(2) symmetry about the origin in the xy-plane. Describe the unitary irreducible representations of O(2). Is this symmetry sufficient to explain the degeneracies?

We see that the Hamiltonian is isotropic because it depends only on length of vectors, not on their direction, $\hat{H} = \hat{H}(\hat{\boldsymbol{X}}^2, \hat{\boldsymbol{P}}^2)$. Thus any transformation of the vectors $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{P}}$ which leave their lengths invariant, leaves the Hamiltonian invariant. These are rotations around the origin along with reflections through mirror lines which goes through the origin. In 2D space, this symmetry group is O(2).

A symmetry of an operator (here the Hamiltonian) leads to degeneracies in its eigenvalues (here energies). The possible number of degeneracies g due to a symmetry is given by the dimensionalities of the irreducible representations of the group corresponding to the given symmetry. If O(2) is the full symmetry of the Hamiltonian, then we would expect representations of O(2) with arbitrarily large dimensionality to exist.

We can decompose the group O(2) into $\mathrm{O}(2) = \mathrm{SO}(2) \otimes \mathrm{Z}_2$, where SO(2) contains all (proper) rotations, which are those which are continuously connected to $\mathbb{1}$, an which therefore have determinant $+1$. Improper rotations are then constructed from rotations along with inversion from Z_2. Now since SO(2) consists solely of rotations around the same axis, this group is abelian, and hence we know that all irreducible representations of SO(2) are 1-dimensional. As rotations are inherently 2-dimensional in the spatial sense, these irreducible representations are necessarily over the field of complex numbers, which gives us the needed two degrees of freedom. An obvious candidate for a representation of SO(2) is then $D^{(m)}(\varphi) = A\mathrm{e}^{\pm \mathrm{i}m\phi}$, where φ is the angle of the rotation. Because products of representation matrices should still be a representation matrix (group element composition is realised as matrix multiplication for representations), we must have $A = 1$. The representation labels (quantum numbers) m can take on any values, as long as $D^{(m)}(\varphi)$ preserves the group structure. Rotations through φ and $\varphi + 2\pi$ are equivalent, and so m are required to be an integer, which also means that the sign in the exponential does not matter:

$$D^{(m)}(\varphi) = \mathrm{e}^{-\mathrm{i}m\phi}\,, \quad m \in \mathbb{Z}\,.$$

Under SO(2) we can then label states by m. There then exists some operator $\hat{J}$ which is the symmetry operator of rotation, the eigenvalue m when acting on a $|m\rangle$ state:

$$\hat{J}\,|m\rangle = m\,|m\rangle\ .$$

Physically we know that the generator of rotations is the angular momentum operator $\hat{J}$. Note that in 2D only a single axis* of rotation is possible, and so be do not have a $\hat{J}^2$ and $\hat{J}_z$ operator, but only a single $\hat{J}$. This $\hat{J}$ is then reminiscent of $\hat{J}_z$, as its eigenvalues are integrally separated.

Now we append the group Z_2 to SO(2) to include inversion $\hat{I}$ as an operator. Doing an inversion flips all directions, and so $\hat{I}$ and $\hat{J}$ anti-commute, yielding

$$\begin{aligned}\hat{J}\big(\hat{I}\,|m\rangle\big) &= -\hat{I}\hat{J}\,|m\rangle \\ &= -m\big(\hat{I}\,|m\rangle\big)\,,\end{aligned}$$

*I guess *plane* of rotation is better terminology, as this axis is in a direction outside of the 2D space.

and so $\hat{I}\,|m\rangle$ is a new eigenvector of $\hat{J}$ with eigenvalue $-m$. Thus

$$\hat{I}\,|m\rangle = |-m\rangle \ .$$

This means that the single vector $|m\rangle$ does not span the vector space required by both $\hat{I}$ and $\hat{J}$ (and hence O(2)), and so we must enlarge this space to $\{|m\rangle\,,\,|-m\rangle\}$, which is then a 2-dimensional irreducible representation. As we have found the degeneracies g larger than this (just pick $n > 1$), the O(2) symmetry of the Hamiltonian cannot explain the degeneracy.

(c) Show that that this Hamiltonian has U(2) symmetry. Hint: Show that the transformation of the annihilation operator

$$\begin{aligned} \hat{a}_1' &= u_{11}\hat{a}_1 + u_{12}\hat{a}_2 \\ \hat{a}_2' &= u_{21}\hat{a}_1 + u_{22}\hat{a}_2 \end{aligned}$$

(and the implied transformation of the creation operator) is a symmetry of the Hamiltonian as long as the u_{ij} satisfy certain criteria. Describe the unitary irreducible representations of U(2). Is this symmetry sufficient to explain the degeneracies?

The transformations on the annihilation operator can be written succinctly as

$$\hat{a}_i \to U_{ik}\hat{a}_k = \hat{a}_i' \,, \tag{2.4}$$

with summation over k implied. The transformations for $i = 1$ and $i = 2$ are really meant to be done simultaneously, i.e. they are not independent transformations. I therefore prefer the notation

$$\hat{\boldsymbol{a}} \to U\hat{\boldsymbol{a}}\,, the$$

where $\hat{\boldsymbol{a}}$ is now a column vector containing $\hat{a}_1$ and $\hat{a}_2$, while for now, U is any complex 2×2 matrix, $U \in \mathbb{C}^{2\times 2}$. Now, if we require $\hat{H}$ to be invariant under this transformation, we see that U must be unitary:

$$\begin{aligned} \hat{H} = \hbar\omega\big(\hat{\boldsymbol{a}}^\dagger\hat{\boldsymbol{a}} + 1\big) &\xrightarrow{\hat{\boldsymbol{a}}\to U\hat{\boldsymbol{a}}} \hbar\omega\big((U\hat{\boldsymbol{a}})^\dagger(U\hat{\boldsymbol{a}}) + 1\big) \\ &= \hbar\omega\big(\hat{\boldsymbol{a}}^\dagger U^\dagger U\hat{\boldsymbol{a}} + 1\big) \\ &\equiv \hat{H} \\ \Rightarrow U^\dagger U = \mathbb{1}\,. \end{aligned}$$

All unitary 2×2 matrices form the group U(2), and so $U \in \mathrm{U}(2)$ and $\hat{H}$ has U(2) symmetry.

For completeness, we might also want to check the invariance of the commutation relations of annihilation and creation operators under the U(2) transformation. From (2.4) we have $\hat{a}_j^\dagger \to (U_{j\ell})^\dagger\hat{a}_\ell^\dagger = (U^\dagger)_{\ell j}a_\ell^\dagger$, and so the commutation relations (2.1) transform as

$$\begin{aligned} [\hat{a}_i,\, \hat{a}_j^\dagger] &\xrightarrow{\hat{a}_i\to U_{ik}\hat{a}_k} \big[U_{ik}\hat{a}_k,\, (U^\dagger)_{\ell j}\hat{a}_\ell^\dagger\big] \\ &= U_{ik}(U^\dagger)_{\ell j}\underbrace{\big[\hat{a}_k,\, \hat{a}_\ell^\dagger\big]}_{\delta_{k\ell}} \end{aligned}$$

$$= U_{ik}(U^\dagger)_{kj}$$
$$= \delta_{ij} \,.$$

Now let us turn to a discussion about the irreducible representations of U(2). As we did in (b), we may decompose U(2) into $\mathrm{U}(2) = \mathrm{SU}(2) \otimes \mathrm{Z}_2$, where now again, SU(2) includes only the proper unitary transformations; those with determinant +1, which are continuously connected to $\mathbb{1}$. We know that the irreducible representations of SU(2) can be labelled by two numbers $j \in {}^1\!/_2\mathbb{N}$ and $m \in \mathbb{Z}$, and that the resulting representation matrices $D^{(j)}$ have dimension $2j+1$. In order to see how many of the $n+1$ degeneracies are due to the U(2) symmetry of $\hat{H}$, we just need to find a connection between j and n.

In [1] they analyze a system of two uncoupled harmonic 1D quantum harmonic oscillators. As evident by the Kronecker deltas in (2.1), (2.2) and (2.3), the two directions of the 2D oscillator is in fact uncoupled, and so we may freely use any result derived in [1]. In particular, they derive our missing relation between j and n (equation 9.17), which in our notation is

$$j = \frac{n_1 + n_2}{2}\,, \qquad m = \frac{n_1 - n_2}{2}\,.$$

This means that for a given j, $-j \leq m \leq j$. The factors of $1/2$ are important, as otherwise m would only be able to take on values in the even numbers: For fixed j, the smallest possible change of a configuration is $|n_1,\, n_2\rangle \to |n_1 \pm 1,\, n_2 \mp 1\rangle$, and so $|n_1 - n_2|$ changes by 2. More importantly for us, $j = 1/2(n_1 + n_2) = 1/2n$ implies that j takes on a single value for each value of n. Remembering that the dimensionality of the irreducible representations for SU(2) (labelled by j and m) is $2j+1$, we indeed have

$$\dim D^{(j)} = n + 1\,,$$

which matches the found degeneracy $n + 1 = g(n) \equiv g(j)$. We conclude then that all of the degeneracy of the Hamiltonian can be explained by the U(2) symmetry.

One might ask why we were able to explain *some* of the degeneracy by considering O(2) symmetry, when *all* of the degeneracy was in the end explained by U(2) symmetry. The answer lies in the fact that O(2) is a subgroup of U(2), which is obvious when considering that the difference between orthogonality $O^\dagger O = \mathbb{1}$ and unitarity $U^\dagger U = \mathbb{1}$ is just a complex conjugation, which does not matter for the real entries in $O \in \mathrm{O}(2)$.

3

In this problem you will explore the homomorphism from SU(2) to SO(3). For every $x \in \mathbb{R}^3$, define a 2×2 matrix by $\tilde{x} = \boldsymbol{x} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the vector of Pauli matrices $\boldsymbol{\sigma} = (\sigma_1,\, \sigma_2,\, \sigma_3)$. This is sometimes called the quaternion representation of the vector.

(a) Show that a general matrix $u \in \mathrm{SU}(2)$ that transforms the quaternion representation of a vector like $u\tilde{x}u^\dagger = \tilde{x}'$ preserves the determinant $\det \tilde{x} = \det \tilde{x}'$. What does that mean?

Since the determinant of a product of matrices is equal to the product of determinants, we have

$$\det\bigl(u\tilde{x}u^\dagger\bigr) = \det u \det \tilde{x} \det u^\dagger$$

$$= \det \tilde{x}\,,$$

where $\det u = 1$ because $u \in \mathrm{SU}(2)$, where the S (special) precisely means determinant $+1$, and $\det u^\dagger = 1$ because $u^\dagger = u^{-1}$, and so $u^\dagger \in \mathrm{SU}(2)$ as well. This means that the transformation obtained by conjugating $\tilde{x}$ with u is a proper similarity transformation, resulting in a new matrix $\tilde{x}'$ which is really just the old $\tilde{x}$ but in a new basis. Thus, the similarity transformation must correspond to a rotation of the original vector $\boldsymbol{x}$. In particular then, the transformation preserves the length of the vector.

(b) Any matrix $u \in \mathrm{SU}(2)$ can be parameterized by three angles $\alpha,\, \beta,\, \gamma \in [0,\, 2\pi)$ like

$$u(\alpha,\, \beta,\, \gamma) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}/2(\alpha+\gamma)} \cos\beta/2 & -\mathrm{e}^{-\mathrm{i}/2(\alpha-\gamma)} \sin\beta/2 \\ \mathrm{e}^{\ \mathrm{i}/2(\alpha-\gamma)} \sin\beta/2 & \mathrm{e}^{\ \mathrm{i}/2(\alpha+\gamma)} \cos\beta/2 \end{pmatrix}.$$

Show that

$$u(\alpha,\beta,\gamma)\tilde{x}u^\dagger(\alpha,\beta,\gamma) = \widetilde{R\boldsymbol{x}}\,, \tag{1}$$

where $\widetilde{R\boldsymbol{x}} = \big(R(\alpha,\beta,\gamma)\boldsymbol{x}\big) \cdot \boldsymbol{\sigma}$ and $R(\alpha,\beta,\gamma) \in \mathrm{SO}(3)$ is the normal zyz-Euler angle matrix.

I will therefore not prove (1) by direct calculation, but rather explain why it has to be true. We have established that the transformation $\tilde{x} \to u\tilde{x}u^\dagger$ is really a rotation, performed not on the vector $\boldsymbol{x}$ but on its quaternion representation $\tilde{x}$. Instead of constructing the quaternion representation and then do the rotation, we might as well rotate the actual vector $\boldsymbol{x}$ and then construct the quaternion representation from this rotated vector. The set of all proper rotations of a 3-vector form the group SO(3), the group of orthogonal matrices with determinant $+1$. Thus there must exist some $R \in \mathrm{SO}(3)$ corresponding to each $u \in \mathrm{SU}(2)$. Both R and u depend on the same number (3) of compact parameters, as they should.

(c) Show that the homomorphism from SU(2) to SO(3) induced by (1) is two-to-one. What is the kernel of the homomorphism?

From (1) it is clear that if u is the matrix in SU(2) that corresponds to $R \in \mathrm{SO}(3)$, so does $-u$. That means that for every rotation R, we have 2 matrices $\pm u$. Thus the correspondence between SU(2) and SO(3) is two-to-one, and so it is a homomorphism but not quite an isomorphism. If it had been, the kernel of the homomorphism would be just $\mathrm{diag}(1,\, 1) \in \mathrm{SU}(2)$, which clearly leaves $\tilde{x}$ invariant and thus corresponds to $\mathrm{diag}(1,\, 1,\, 1) \in \mathrm{SO}(3)$. However, as already stated, both of $\pm u$ corresponds to the same rotation in SO(3), and so the kernel also contains $\mathrm{diag}(-1,\, -1)$.

We ought to prove the claim that if $u \in \mathrm{SU}(2)$, so is $-u$. First, $-u$ is unitary; $(-u)(-u)^\dagger = uu^\dagger = \mathbb{1}_2$, where $\mathbb{1}_2$ is a 2×2 unit matrix. Second, it has determinant $+1$; $\det(-u) = \det(-\mathbb{1}_2)\det(u) = (-1)^2(+1) = +1$. An interesting detail now appears: For $-u$ to be in the special unitary group, the group has to be of even dimension. Thus we do not expect to find similar double-coverings in e.g. SU(3).

(d) Why do we use SU(2) instead of SO(3) when studying rotational invariance in quantum mechanics? What does it give us?

Classical vectors like $\boldsymbol{x}$ transform under SO(3) when rotated, and working with SU(2) when dealing with these would be pointless. Physically, we expect certain quantities to be conserved under rotation, e.g. the total angular momentum. In quantum mechanics however, these quantities are promoted to operators, and it is these operators which transform under rotation.

We might imagine the rotation as being applied to all states (Schrödinger picture), $|\psi\rangle \to U^\dagger |\psi\rangle$, where $|\psi\rangle = |\psi(\boldsymbol{x})\rangle = |\psi(\tilde{x})\rangle$. Requiring that some observable be invariant under rotation means preservation of matrix elements, e.g. $\langle\phi|\hat{J}^2|\psi\rangle \to \langle\phi|U\hat{J}^2U^\dagger\psi|\rangle$. Instead of acting on the states, we see that the transformation can be applied to the operator (Heisenberg picture): $\hat{J}^2 \to U\hat{J}^2U^\dagger$. Again, we see that both of $\pm U$ preserves the matrix element.

Now removing the operator and just consider $\langle\phi|\psi\rangle$. We would always require this to be invariant under a full 2π rotation. Again, this still allows for both $|\psi\rangle$ and $\langle\phi|$ to pick up a sign*, which we might think of as being do to some $U^\dagger$, U acting on $|\psi\rangle$, $|\phi\rangle$. Thus, we can imagine a state which is only 2π-periodic up to a sign, but actually only completely 4π-periodic. It turns out that these states transform under half-integer j representations of SU(2), in particular $j = 1/2$, which we call spinors. Such "strange" objects are not only mathematically possible, but as nature would have it fully realised as fermions.

6

The proper Euclidean group in two dimensions E_2^+ is the semidirect product $\mathrm{SO}(2) \ltimes \mathrm{T}_2$ of rotations and translations in a plane. A general element is denoted $(R,\, \boldsymbol{a})$, where $R \in \mathrm{SO}(2)$ and $\boldsymbol{a} \in \mathrm{T}_2 \cong \mathbb{R}^3$. I presume what is really meant is $\mathbb{R}^2$?

(a) By constructing the faithful representation of E_2^+ on vectors $\boldsymbol{x} \in \mathbb{R}^2$ by

$$T(R,\, \boldsymbol{a})\boldsymbol{x} = R\boldsymbol{x} + \boldsymbol{a}$$

prove the group composition rule $(R',\, \boldsymbol{a}') \circ (R,\, \boldsymbol{a}) = (R'R,\, R'\boldsymbol{a} + \boldsymbol{a}')$. What is the inverse of $(R,\, \boldsymbol{a})$?

Applying two Euclidean transformations successively, we get

$$\begin{aligned} T(R',\, \boldsymbol{a}')T(R,\, \boldsymbol{a})\boldsymbol{x} &= T(R',\, \boldsymbol{a}')(R\boldsymbol{x} + \boldsymbol{a}) \\ &= R'(R\boldsymbol{x} + \boldsymbol{a}) + \boldsymbol{a}' \\ &= R'R\boldsymbol{x} + (R'\boldsymbol{a} + \boldsymbol{a}') \\ \Rightarrow (R',\, \boldsymbol{a}') \circ (R,\, \boldsymbol{a}) &= (R'R,\, R'\boldsymbol{a} + \boldsymbol{a}')\,. \end{aligned}$$

One might naïvely expect that the inverse of $(R,\, \boldsymbol{a})$ is $(R^{-1},\, -\boldsymbol{a})$. As Euclidean transformations are carried out by *first* rotating $\boldsymbol{x}$, *then* translating it, $R\boldsymbol{x} + \boldsymbol{a} \neq R(\boldsymbol{x} + \boldsymbol{a})$, this

*Or more generally, two opposite phases. As these must be opposite, they do not really have a phase degree of freedom, and so we might as well just work with a sign change.

does not work as we would have to translate by $-\boldsymbol{a}$ first, then rotate by R^{-1} to cancel the original transformation. To find the true inverse transformation, we simply set the result of the composition rule equal to the identity transformation $E_2^+ \ni e = (\mathbb{1},\, \boldsymbol{0})$, yielding

$$
\begin{aligned}
(R'R,\, R'\boldsymbol{a} + \boldsymbol{a}') = (\mathbb{1},\, \boldsymbol{0}) \Rightarrow &\begin{cases} R' = R^{-1}\,, \\ \boldsymbol{a}' = -R'\boldsymbol{a} = -R^{-1}\boldsymbol{a} \end{cases} \\
\Rightarrow (R,\, \boldsymbol{a})^{-1} = (R^{-1},\, -R^{-1}\boldsymbol{a})\,.
\end{aligned}
$$

(b) A (properly unitized) matrix representation of E_2^+ that acts on extended vectors $\tilde{x} = (x_1,\, x_2,\, 1)^\top$ is given by

$$
E\big(R(\theta),\, \boldsymbol{a}\big) = \begin{pmatrix} \cos\theta & -\sin\theta & a_1/\ell \\ \sin\theta & \cos\theta & a_2/\ell \\ 0 & 0 & 1 \end{pmatrix}
$$

where ℓ is an arbitrary length scale and $\theta \in [0, 2\pi)$. Define J as the generator associated with parameter θ and P_1 and P_2 as the generators with parameters a_1 and a_2. Find the matrix form of these generators.

Consider the subgroup $\mathrm{SO}(2) \subset \mathrm{E}_2^+$ for a moment. Elements of this group are $R(\theta)$, and so it is this group that J generates. We can write a general element via the expansion

$$
R(\theta) = \sum_{n=0}^{\infty} \frac{(-\mathrm{i}\theta J)^n}{n!} \tag{6.1}
$$

which we recognize as the Taylor expansion of the matrix exponential of $-\mathrm{i}\theta J$, where the $-\mathrm{i}$ is just a convention. For $\theta \to 0$ this gives the identity $\mathbb{1}_3$. Since all of SU(2) is *continuously* connected to this identity *and* we only have a single rotation axis, we are guaranteed that a single J can generate all $R(\theta)$ in this manner. We can thus imagine the topology of SO(2) as a circle, naturally parameterized by θ, and where the generator J "points" along the $\hat{\theta}$ direction.

If we differentiate (6.1) with respect to θ and then evaluate the equation in $\theta = 0$, only the $n = 1$ term in the exponential expansion will survive, yielding

$$
\left.\frac{\mathrm{d}R(\theta)}{\mathrm{d}\theta}\right|_{\theta=0} = -\mathrm{i}J\,. \tag{6.2}
$$

Enlarging the group to the entirety of E_2^+, we now calculate J explicitly from $E \in \mathrm{E}_2^+$:

$$
J = \mathrm{i}\left.\frac{\mathrm{d}E\big(R(\theta),\, \boldsymbol{a}\big)}{\mathrm{d}\theta}\right|_{\theta=0} = \mathrm{i}\left.\begin{pmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\right|_{\theta=0} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Note that this J, the generator of rotations in E_2^+ is slightly different from the previous J, the generator of rotations in SO(2); the first is the latter with one additional null row and column added. Similarly for P_1 and P_2, we have

$$
P_1 = \mathrm{i}\left.\frac{\mathrm{d}E\big(R(\theta),\, \boldsymbol{a}\big)}{\mathrm{d}a_1}\right|_{a_1=0} = \mathrm{i}\left.\begin{pmatrix} 0 & 0 & 1/\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right|_{a_1=0} = \begin{pmatrix} 0 & 0 & \mathrm{i}/\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$P_2 = \mathrm{i}\,\frac{\mathrm{d}E\big(R(\theta),\,\boldsymbol{a}\big)}{\mathrm{d}a_2}\bigg|_{a_2=0} = \mathrm{i}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1/\ell\\ 0 & 0 & 0\end{pmatrix}_{a_2=0} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & \mathrm{i}/\ell\\ 0 & 0 & 0\end{pmatrix}.$$

For completeness, let us (re)construct the pure rotations and pure translations of the matrix representation by means of (6.2), that is by exponentiating the generators. For rotations, we have

$$\begin{aligned}
E\big(R(\theta),\,\boldsymbol{0}\big) &= \sum_{n=0}^{\infty}\frac{(-\mathrm{i}\theta J)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{\theta^n}{n!}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}^n\\
&= \mathbb{1}_3 + \sum_{n\in 2\mathbb{N}^+}\frac{\theta^n}{n!}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}^n + \sum_{n\in 2\mathbb{N}+1}\frac{\theta^n}{n!}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}^n,
\end{aligned}$$

where the infinite sum has been split into three parts; an identity matrix $\mathbb{1}_3 = \mathrm{diag}(1,\,1,\,1)$, a sum over all even terms excluding zero and a sum over all odd terms. This split is useful because of the following property of the matrix $(-\mathrm{i}J)$ appearing in the sums:

$$\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}^n = \begin{cases}\mathbb{1}_3 & \text{if } n = 0\,,\\ (-1)^{n/2}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} & \text{if } n\in 2\mathbb{N}\,,\\ (-1)^{\frac{n+1}{2}}\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} & \text{if } n\in 2\mathbb{N}+1\,,\end{cases}$$

where the special case $n = 0$ is the reason why $\mathbb{1}_3$ has been separated out of the sums. We can then take the matrices out of the sums, change the summation variable and recognize these sum for what they are:

$$\begin{aligned}
E\big(R(\theta),\,\boldsymbol{0}\big) &= \mathbb{1}_3 + \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\sum_{n\in 2\mathbb{N}^+}\frac{(-1)^{n/2}\theta^n}{n!} + \begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\sum_{n\in 2\mathbb{N}+1}\frac{(-1)^{\frac{n+1}{2}}\theta^n}{n!},\\
&= \mathbb{1}_3 + \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^n\theta^{2n}}{(2n)!}}_{\cos\theta-1} + \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\underbrace{\sum_{n=0}^{\infty}\frac{(-1)^n\theta^{2n+1}}{(2n+1)!}}_{\sin\theta},\\
&= \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}.
\end{aligned}$$

We have thus successfully constructed a general pure rotation by exponentiation of the generator J. For the generators of translation P_i this exponentiation is much easier, as we see that $P_i^2 = \mathbb{0}_3$ for both $i = 1$ and $i = 2$, thus truncating the infinite sums. The pure translations are then

$$\begin{aligned} E(\mathbb{1}_3, \boldsymbol{a}) &= \sum_{n=0}^{\infty} \frac{(-\mathrm{i}\boldsymbol{a} \cdot \boldsymbol{P})^n}{n!} \\ &= \sum_{n=0}^{1} \frac{(-\mathrm{i}\boldsymbol{a} \cdot \boldsymbol{P})^n}{n!} \\ &= \mathbb{1}_3 - \mathrm{i}\boldsymbol{a} \cdot \boldsymbol{P} \\ &= \begin{pmatrix} 1 & 0 & a_1/\ell \\ 0 & 1 & a_2/\ell \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

where $\boldsymbol{P} \equiv (P_1, P_2)$. Note that we simply exponentiated both P_1 and P_2 together. This is allowed because of the property of the exponential $\exp(A)\exp(B) = \exp(A + B)$. This however is only true if A and B commute, which for P_1 and P_2 means that their directions of translations must be perpendicular. We can think of T_2 as being made up of $\mathrm{T}_2 = \mathrm{T}_1 \otimes \mathrm{T}_1$, and associate one P_i with each T_1.

(c) Show that the matrix forms for J, P_1 and P_2 satisfy the relations

$$[P_1, J] = -\mathrm{i}\hbar P_2, \quad [P_2, J] = \mathrm{i}\hbar P_1, \quad [P_1, P_2] = \mathbb{0}_3\,.$$

Note that the arbitrary length scale does not appear.

Direct calculation gives

$$\begin{aligned} [P_1, J] &= \overbrace{\begin{pmatrix} 0 & 0 & \mathrm{i}/\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{\mathbb{0}_3} - \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{i}/\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\ell \\ 0 & 0 & 0 \end{pmatrix} \\ &= -\mathrm{i}P_2\,, \end{aligned}$$

$$\begin{aligned} [P_2, J] &= \overbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}/\ell \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{\mathbb{0}_3} - \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}/\ell \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & -1/\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathrm{i}P_1\,, \end{aligned}$$

$$\begin{aligned}[P_1, P_2] &= \overbrace{\begin{pmatrix} 0 & 0 & \mathrm{i}/\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}/\ell \\ 0 & 0 & 0 \end{pmatrix}}^{\mathbb{0}_3} - \overbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i}/\ell \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & \mathrm{i}/\ell \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{\mathbb{0}_3} \\ &= \mathbb{0}_3\,.\end{aligned}$$

The arbitrary length scale ℓ do indeed not enter in the commutation relations. I have defined J in natural units (the choice was made back in (6.1)), which is why $\hbar$ is missing from my expressions. We can "fix" this by letting $J \to \hbar J$ and simply remember to divide the new dimensional J by $\hbar$ when it appears in expressions such as (6.1) where a dimensionless argument is required.

(d) Using these commutation relations (and not the matrices), show that $P_1^2 + P_2^2$ is an invariant. What does this mean? Can you come up with a three-dimensional Hamiltonian that has E_2^+ symmetry?

That an operator (e.g. $P_1^2 + P_2^2$) is invariant under E_2^+ means that its eigenvalues are the same for some vector before and after any E_2^+ transformation. This can be checked by computing these values for each of the infinitely many elements in E_2^+. This invariance can also be expressed as a vanishing commutator between the operator in question and each of the elements in E_2^+. Though not themselves elements of E_2^+, the three generators contains the entire algebraic structure of the group, and so it suffices to check commutativity with these three elements.

Let us compute the commutator of $P_1^2 + P_2^2$ and J:

$$\begin{aligned}\big[P_1^2 + P_2^2,\, J\big] &= \big[P_1^2,\, J\big] + \big[P_2^2,\, J\big] \\ &= P_1\big[P_1,\, J\big] + \big[P_1,\, J\big]P_1 + P_2\big[P_2,\, J\big] + \big[P_2,\, J\big]P_2 \\ &= -\mathrm{i}P_1P_2 - \mathrm{i}P_2P_1 + \mathrm{i}P_2P_1 + \mathrm{i}P_1P_2 \\ &= \mathbb{0}_3\,.\end{aligned}$$

As the generators of translation mutually commute, we immediately have

$$\big[P_1^2 + P_2^2,\, P_i\big] = \mathbb{0}_3\,.$$

Thus we can conclude that $P_1^2 + P_2^2$ is conserved under E_2^+. Since this is really the squared length of the vector $\boldsymbol{P}$, I would suspect that it is even conserved under E_2. Because P_i is the generator of translation, $P_1^2 + P_2^2$ is really the operator for the total squared momentum. For a Hamiltonian with E_2^+ as a symmetry group, the total squared momentum is then a conserved quantity, in accordance with Noether's theorem.

The simplest such E_2^+ symmetric two-dimensional Hamiltonian would be the free Hamiltonian in 2D space, which naturally have to respect spatial (Euclidean) symmetry:

$$\hat{H}_0 = \frac{\hat{\boldsymbol{P}}^2}{2m}$$

where $\hat{\boldsymbol{P}}^2$ is precisely the operator corresponding to $P_1^2 + P_2^2$. If we consider the free Hamiltonian in 3-space, the plane in which P_1 and P_2 operate can now be rotated out in the third dimension, and so $P_1^2 + P_2^2$ becomes just a projection of the true squared momentum $P_1^2 + P_2^2 + P_3^2$. We could alter the Hamiltonian in such a manner as to restrict the momentum to the plane, but then I would not really call it a three-dimensional Hamiltonian any more.

Let us go back to 2-space and a potential $V(\boldsymbol{x})$. As for the free Hamiltonian, this can only depend on the length of the vector due to our demand of rotational symmetry; $V(\boldsymbol{x}) = V(\boldsymbol{x}^2) = V(|\boldsymbol{x}|)$. Because $\boldsymbol{x}$ is an actual spatial vector, we now also have to take translational symmetry into consideration. If $\boldsymbol{x}$ is "attached" to the background metric (i.e. it points from the origin to some particle), this breaks the translational symmetry. We are thus only allowed to use spatial vectors $\boldsymbol{x}$ *between* two objects/coordinates, and again only the length of this; $V(\boldsymbol{x}) = V(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)$. Brining V into 3-space again means that we now have T_3 symmetry and not T_2 symmetry. We could however retain the T_2 symmetry in 3-space by making the third axis homogeneous (e.g. using infinite rods instead of particles). As for rotations, we face the same problems as with $\hat{\boldsymbol{P}}^2$, namely that $\boldsymbol{x}_1 - \boldsymbol{x}_2$ can be rotated to have a component in the third direction. I do not see any way out of this, except effectively restricting the system to a plane.

References

[1] Jun John Sakurai. *Modern quantum mechanics.* Addison-Wesley, Reading, MA, 2. edition, 1994.