## Calculation of a function of complex variable by solving differential equations

Let us e.g. calculate $\exp(z)$ from the differential equation $\frac{du}{dt} = u$ with the boundary condition $u(0) = 1$. Introducing the real parameter $s \in [0, 1]$ and making a substitution $t = sz$, we get $u'(s) = zu(s)$, $u(0) = 1$, where the prime denotes $\frac{d}{ds}$. To avoid complex arithmetics we introduce $v = \Re(u), w = \Im(u)$ and get a system of two real ordinary differential equations (with $z = x+iy$)

$$\begin{cases} v'(s) = xv(s) - yw(s), \\ w'(s) = xw(s) + yv(s), \\ v(0) = 1, \\ w(0) = 0, \end{cases} , \tag{1}$$

the solution to which gives the sought $\exp(z)$: $\Re(\exp(z)) = v(1)$, $\Im(\exp(z)) = w(1)$.

This system can be solved with our routines rkdriver/pcdriver. For larger $z$ one can first use

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2), \tag{2}$$

$$\exp(n) = e^n,\ \exp(i\pi/2) = i. \tag{3}$$

## Calculation of a function of complex variable by integration

Let us calculate e.g. $\arctan(z)$ using the integral representation

$$\arctan(z) = \int_0^z \frac{dt}{1+t^2}. \tag{4}$$

The integration path can be taken as a straight line from 0 to $z = x + iy$ as $t = sz$, where $s \in [0, 1]$ is a real parameter,

$$\arctan(z) = z\int_0^1 \frac{ds}{1+(sz)^2}. \tag{5}$$

one can avoid using complex arithmetics by calculating separately the real and imaginary parts of the integral with our routine adapt. If $|z| > 1$ one should perhaps instead use

$$\arctan(z) = \frac{\pi}{2} - \int_z^\infty \frac{dt}{1+t^2}. \tag{6}$$