1 Feynman's path integral method in quantum mechanics

Consider a one-dimentional motion of a particle with mass m, coordinate x and Lagrangian $L(x, \dot{x})$. Suppose $|x_a\rangle$ and $|x_b\rangle$ are the eigenfunctions of the coordinate operator with the eigenvalues x_a and x_b. Suppose $U(t_b, t_a) = e^{-\frac{i}{\hbar}H(t_b - t_a)}$ is the time-evolution operator,

$$\Psi(x, t_b) = U(t_b, t_a)\Psi(x, t_a). \tag{1}$$

Then the propagator

$$K_{x_b x_a}(t_b, t_a) \equiv \langle x_b|U(t_b, t_a)|x_a\rangle \tag{2}$$

is intuitively defined as the Feynman's integral over all classical paths $x(t)$ connecting (x_a, t_a) and (x_b, t_b),

$$K_{x_b x_a}(t_b, t_a) = \int_{x(t)} D[x(t)] e^{\frac{i}{\hbar}S[x(t)]} \tag{3}$$

where $D[x(t)]$ is a certain functional meausure, and $S[x(t)] = \int dt L(x, \dot{x})$ is the action calculated along the path $x(t)$.

The stationary phase approach says that when $\hbar \to 0$ only the stationary path, that is where $\frac{\delta S}{\delta x} = 0$, gives nonvanishing contribution to the integral. This is apparently the classical variational principle which defines the classical path.

1.1 Time discretization

Let us discretize time in N equidistant slices $t_0 = t_a, t_1, ..., t_N = t_b$, with $\Delta t = \frac{(t_b - t_a)}{N}$

$$K_{x_b x_a}(t_b, t_a) \approx \left(\frac{m}{2\pi i\hbar\Delta t}\right)^{\frac{N}{2}} \int dx(t_1)...dx(t_{N-1}) \times \prod_n e^{\frac{i}{\hbar}S_n} \tag{4}$$

where the short time action is defined using the *midpoint rule*,

$$S_n = \Delta t L\left[\frac{x_n + x_{n-1}}{2}, \frac{x_n - x_{n-1}}{\Delta t}\right] \tag{5}$$

The (broken-line) path $x(t)$ is now defined by the set of values $x(t_1), x(t_2), ..., x(t_{N-1})$

The normalization constant $\left(\frac{m}{2\pi i\hbar\Delta t}\right)^{1/2}$ comes from considering a short-time propogator for a free particle, which can be calculated analytically.

1.2 Space discretization

With the space discretized also, $x = x_0, x_1, ..., x_D$, the short-time propagator becomes a complex matrix

$$K_{ij}(\Delta t) = \sqrt{\frac{m}{2\pi i\hbar\Delta t}}\, e^{\frac{i\Delta t}{\hbar}L\left[\frac{x_j + x_i}{2}, \frac{x_j - x_i}{\Delta t}\right]} \tag{6}$$

and consequently the finite time propagator $K_{ij}(t) = \langle x_i|U(t)|x_j\rangle$ becomes a product of short-time propagators, $K(t) = \Delta x^{N-1}K(\Delta t)^N$.

1.3 Real time propagation

The energy levels can be extracted by a Fourier transform of the trace of the propagator:

$$\text{trace}(U(t)) = \sum_n \langle n|e^{-\frac{i}{\hbar}Ht}|n\rangle = \sum_n e^{-\frac{i}{\hbar}E_n t} \tag{7}$$