Classical radiation theory

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General description of radiation

Let us consider a particle with charge e traveling inside a confined region in a trajectory $\mathbf{r}_0(t_0)$ with velocity $\mathbf{v}(t_0)$.

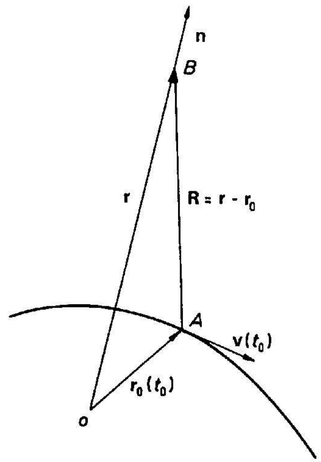

Figure 1: Radiation of a particle moving along the trajectory $\mathbf{r}_0(t_0)$, which at the time t_0 is at point A, is observed at point B with coordinate $\mathbf{r}$ at the time t.

The electromagnetic field created by the particle will be considered at a distance r large compared with to the system dimension. In this case one can choose the origin of coordinate inside the system and make the expansion of all values over r_0/r

$$R = |\mathbf{r} - \mathbf{r}_0| \simeq r \quad 1 - \frac{\mathbf{n}\cdot\mathbf{r}_0(t_0)}{r} \quad , \quad \mathbf{n} = \frac{\mathbf{r}}{r}. \tag{1}$$

The time in the observation point t is connected with the "particle time" t_0 by the following retardation relation

$$t_0 = t - R(t_0) \simeq t - r + \mathbf{n}\cdot\mathbf{r}_0(t_0) \tag{2}$$

At large distance from the particle, where one should keep only the slowly decreasing terms $\sim 1/r$, the fields have the following form

$$\mathbf{A} = \frac{e}{r}\frac{\mathbf{v}}{(1-\mathbf{n}\cdot\mathbf{v})}, \quad \mathbf{E} = \frac{e}{r}\frac{1}{(1-\mathbf{n}\cdot\mathbf{v})^3}(\mathbf{n}\times(\mathbf{n}-\mathbf{v})\times\mathbf{w})), \quad \mathbf{H} = \mathbf{n}\times\mathbf{E}, \tag{3}$$

where $\mathbf{v}$ is the particle's velocity, $\mathbf{w} = \dot{\mathbf{v}}$. In this case the values in the right-hand part are taken at the time moment t_0 defined by Eq.(2). These fields satisfy the following relations

$$\mathbf{E} \cdot \mathbf{H} = \mathbf{E} \cdot \mathbf{n} = \mathbf{H} \cdot \mathbf{n} = 0, \quad \mathbf{E} \times \mathbf{H} = E^2 \mathbf{n} = H^2 \mathbf{n}, \tag{4}$$

i.e. $\mathbf{n}, \mathbf{E}, \mathbf{H}$ form the right-hand triplet of orthogonal vectors that is characteristic for the plane electromagnetic wave propagating at distances which are large comparing with the system dimensions and the lengths of radiated waves (in so called radiation wave zone). Since $\mathbf{E} = \mathbf{H} \times \mathbf{n}$ and $\mathbf{H} = \mathrm{rot}\mathbf{A}$ it is clear that for the description of the electromagnetic field in the wave zone it is sufficient to know the vector-potential $\mathbf{A}(\mathbf{r}, t)$.

The radiation intensity

The radiation intensity dI into the element of solid angle $d\Omega$ is defined as an amount of energy passing per unit time through the element of spherical surface $dS = \mathbf{r}^2 d\Omega$ with a center in the origin of coordinates. It is expressed in terms of Poynting's vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}/4\pi = E^2\mathbf{n}/4\pi$:

$$dI = E^2 r^2 d\Omega/4\pi \tag{5}$$

If we are interested in the complete radiation for the whole time of the charge motion then we have to integrate the intensity over time. In this case, one should take into account that the functions in the right-hand side depends on the time t_0. Taking into account that (see Eq.(2))

$$dt = (\partial t/\partial t_0)dt_0 = (1 - \mathbf{n} \cdot \mathbf{v}))dt_0 \tag{6}$$

we get for the energy radiated by a particle per unit time of the "particle own time"()at the time of radiation) t_0:

$$\frac{d\varepsilon(\mathbf{n})}{dt_0} = \frac{e^2}{4\pi}\frac{1}{(1-\mathbf{n}\cdot\mathbf{v})^5}[2(\mathbf{w}\cdot\mathbf{v})(\mathbf{n}\cdot\mathbf{w})(1-\mathbf{n}\cdot\mathbf{v})$$
$$+\mathbf{w}^2(1-\mathbf{n}\cdot\mathbf{v})^2 - (1-v^2)(\mathbf{n}\cdot\mathbf{w})^2]d\Omega \tag{7}$$

The integration of this expression over angle makes it convenient to carry out in the tensor form using the invariance with respect to three-dimensional rotation:

$$\int \frac{n_i n_k}{(1-\mathbf{n}\cdot\mathbf{v})^5}d\Omega = \frac{4\pi}{3(1-v^2)^4}[\delta_{ik}(1-v^2) + 6v_i v_k];$$
$$\int \frac{n_i}{(1-\mathbf{n}\cdot\mathbf{v})^4}d\Omega = \frac{16\pi v_i}{3(1-v^2)^3}; \quad \int \frac{1}{(1-\mathbf{n}\cdot\mathbf{v})^3}d\Omega = \frac{4\pi}{(1-v^2)^2} \tag{8}$$

Substituting this integrals into Eq.(7) we get for the radiation intensity (the energy radiated per unit time)

$$\frac{d\varepsilon}{dt_0} = \frac{2}{3}\frac{e^2}{(1-v^2)^3}[(\mathbf{w}\cdot\mathbf{v})^2 + w^2(1-v^2)]. \tag{9}$$

Taking into account that the particle acceleration in the external electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ is

$$\mathbf{w} = \frac{e}{m}(1-v^2)^{1/2}[\mathbf{E} + \mathbf{v}\times\mathbf{H} - \mathbf{v}(\mathbf{v}\cdot\mathbf{E})] \tag{10}$$

one can present the radiation intensity in terms of external electric and magnetic fields

$$\frac{d\varepsilon}{dt_0} = \frac{2}{3}\frac{e^4}{m^2(1-v^2)}\{[\mathbf{E} + \mathbf{v}\times\mathbf{H}]^2 - (\mathbf{v}\cdot\mathbf{E})^2\}. \tag{11}$$

Let us consider the radiation in two particular cases.

1. The particle velocity and acceleration are parallel. Then from Eq.(9) we have

$$\frac{d\varepsilon}{dt_0} = \frac{2}{3}\frac{e^2w^2}{(1-v^2)^3}. \tag{12}$$

The motion in the electric field $(\mathbf{H} = 0)$, if $\mathbf{v}||\mathbf{E}$, is an example of such motion. From Eq.(11) one has

$$\frac{d\varepsilon}{dt_0} = \frac{2}{3}\frac{e^4E^2}{m^2}. \tag{13}$$

2. The particle velocity and acceleration are perpendicular. Then from Eq.(9) we have

$$\frac{d\varepsilon}{dt_0} = \frac{2}{3}\frac{e^2w^2}{(1-v^2)^2}. \tag{14}$$

An example of such motion is a motion in the magnetic field $(\mathbf{E} = 0)$. If $\mathbf{v} \perp \mathbf{H}$ one has

$$\frac{d\varepsilon}{dt_0} = \frac{2}{3}\frac{e^4H^2v^2}{m^2(1-v^2)}. \tag{15}$$

Spectral distribution

Consider the wave field expanded into the Fourier integral

$$\mathbf{A}(\omega) = \frac{e}{r} e^{i\omega r} \int \mathbf{v}(t) e^{i(\omega t - \mathbf{k}\mathbf{r}_0(t))} dt, \tag{16}$$

where ω is the frequency and $\mathbf{k} = \mathbf{n}\omega$ is the wave vector. Taking into account that $\mathbf{E} = -\dot{\mathbf{A}},\ \dot{\mathbf{A}} = \partial \mathbf{A}/\partial t,\ \mathbf{H} = \mathbf{n} \times \mathbf{E}$ we can write

$$\mathbf{H} = (\dot{\mathbf{A}} \times \mathbf{n}), \quad \mathbf{E} = (\dot{\mathbf{A}} \times \mathbf{n}) \times \mathbf{n}. \tag{17}$$

The explicit form of fields is

$$\mathbf{H}(\omega) = \frac{ie\omega}{r} e^{i\omega r} \int e^{i(\omega t - \mathbf{k}\mathbf{r}_0(t))} \mathbf{n} \times d\mathbf{r}_0,$$

$$\mathbf{E}(\omega) = \frac{ie\omega}{r} e^{i\omega r} \int (\mathbf{v} - \mathbf{n}) e^{i(\omega t - \mathbf{k}\mathbf{r}_0(t))} dt \tag{18}$$

The most interesting is knowing the total amount of energy emitted for the total time of the process. If $d\varepsilon(\mathbf{n}, \omega)$ is an energy emitted into the element of the solid angle $d\Omega$ in the form of waves in the frequency interval $\omega, \omega + d\omega$ then an analogue of Eq.(5) has the form

$$d\varepsilon(\mathbf{n}, \omega) = |\mathbf{E}(\omega)|^2 \frac{d\omega d\Omega}{4\pi^2} r^2 = |\mathbf{E}(\omega)|^2 \frac{d^3k}{4\pi^2\omega^2} r^2 \tag{19}$$

Substituting Eq.(18) into the expression we have

$$d\varepsilon(\mathbf{n},\omega) = e^2 \int\int [\mathbf{v}(t_1) - \mathbf{n}][\mathbf{v}(t_2) - \mathbf{n}] e^{i(\omega(t_1-t_2)-\mathbf{k}(\mathbf{r}_0(t_1)-\mathbf{r}_0(t_2))} dt_1 dt_2 \frac{d^3k}{(2\pi)^2}. \tag{20}$$

Here $\mathbf{r}(t_1))$ and $\mathbf{r}(t_2))$ are position of the particle on the trajectory at the time moments t_1 and t_2 respectively. Integrating the terms containing $\mathbf{v}(t_1)\mathbf{n}$ and $\mathbf{v}(t_2)\mathbf{n}$ by parts we obtain

$$d\varepsilon(\mathbf{n},\omega) = e^2 \int\int [\mathbf{v}(t_1)\mathbf{v}(t_2) - 1] e^{i(\omega(t_1-t_2)-\mathbf{k}(\mathbf{r}_0(t_1)-\mathbf{r}_0(t_2))} dt_1 dt_2 \frac{d^3k}{(2\pi)^2}. \tag{21}$$

Polarization

The electromagnetic field of a wave is characterized by the vectors $\mathbf{E}$ and $\mathbf{H}$ which are perpendicular to the direction of propagation $\mathbf{n}$. For definiteness the vector $\mathbf{E}$ is usually chosen to characterize the important property of electromagnetic wave called the polarization. One can project the Fourier component of an electric field onto two mutually orthogonal and, generally speaking, complex vectors $\mathbf{e}_\lambda$ $(\lambda = 1, 2)$ of unit length which is $\mathbf{e}_\lambda \perp \mathbf{n}$:

$$\mathbf{e} = e_1 \mathbf{e}_{(1)} + e_2 \mathbf{e}_{(2)} = \mathbf{e}_{(1)} \cos\alpha + \mathbf{e}_{(2)} \sin\alpha e^{i\beta}. \qquad (22)$$

The description of polarization is the same within the classical and quantum theory. In the classical theory the quantities e_1 and e_2 are proportional to the magnitudes of components of the wave electric field (a square root from the energy flux density is usually removed as a common factor). In the quantum theory the quantities $|e_1|^2$ and $|e_2|^2$ are the probabilities of the photon being polarized along $\mathbf{e}_{(1)}$ and $\mathbf{e}_{(2)}$, correspondingly. At $\beta = 0$ the wave is linearly polarized at the angle α to $|e_1|^2$. At $\beta = \pm\pi/2$ and $\alpha = \pi/4$ the wave is right(R) or left(L) circularly polarized.

Physical quantities contain the polarization of the wave (or a photon) in the combination $\varrho_{ik} = e_i(k,\lambda)e_k^*(k,\lambda)$. Using the unit matrix I and the Pauli matrices σ_k one can represent the density matrix in the form

$$\varrho_{ik} = \frac{E_i(\omega)E_k^*(\omega)}{\mathbf{E}\mathbf{E}^*} = \frac{1}{2}\left[\delta_{ik} + \sum_n^3 (\sigma_n)_{ik}\xi_n\right] = \frac{1}{2}(1 + \boldsymbol{\sigma}\boldsymbol{\xi})_{ik} \tag{23}$$

As ϱ_{ik} is an Hermitian matrix, the parameters ξ_n are real. It's evident that diagonal elements are real and $\varrho_{11} + \varrho_{22} = 1$. For pure state $\det\varrho = 0$ and one has from Eq.(23)

$$\det \varrho = \frac{1}{4} \det \begin{pmatrix} 1+\xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 1-\xi_3 \end{pmatrix} = \frac{1}{4}(1-\boldsymbol{\xi}^2) = 0. \qquad (24)$$

So, the polarization properties of a monochromatic plane wave which,by definition, is completely polarized, can be described by the three real parameters ξ_n called the Stokes parameters and satisfying the condition $\boldsymbol{\xi}^2 = \sum_n^3 \xi_n^2 = 1$.

To elucidate the physical meaning of Stokes parameters let us consider examples. Let the wave propagates along x_3 axis. The wave is circularly polarized is $\mathbf{e}_\pm = (\mathbf{e}_{(1)} \pm i\mathbf{e}_{(2)})/\sqrt{2}$ where the signs $+$ (-) corresponds to the right (left) polarization. In this case Eq.(23) takes a form

$$\varrho = \frac{1}{2} \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix} = \frac{1}{2}(I \pm \sigma_2), \qquad (25)$$

i.e. at circular (right or left) polarization, the Stokes parameters are $\xi_1 = \xi_3 = 0, \xi_2 = \pm 1$.

At linear polarization along x_1 axis (along $\mathbf{e}_{(1)}$) we have from Eq.(23)

$$\varrho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(I + \sigma_3), \quad \text{or} \quad \varrho = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}(I - \sigma_3). \tag{26}$$

In this case $\xi_1 = \xi_2 = 0, \xi_3 = \pm 1$ (the signs $+$ or - correspond to the polarization along x_1- or x_2-axes).

Consider, finally, the linear polarization at the angle $\pm\pi/4$ to $\mathbf{e}_{(1)}$, when we find from Eq.(23)

$$\varrho = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix} = \frac{1}{2}(I \pm \sigma_1), \tag{27}$$

i.e. the Stokes parameters of the wave linearly polarized at the angle $\pm\pi/4$ to the x_1-axis are $\xi_2 = \xi_3 = 0, \xi_1 = \pm 1$. For linear polarization at the angle φ to the x_1-axis one has

$$\varrho = \frac{1}{2}\begin{pmatrix} 1+\cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & 1-\cos 2\varphi \end{pmatrix} = \frac{1}{2}(I + \sigma_3 \cos 2\varphi + \sigma_1 \sin 2\varphi), \qquad (28)$$

or $\xi_1 = \sin 2\varphi$, $\xi_3 = \cos 2\varphi$ and $\xi_1^2 + \xi_3^2 = 1$. i.e. at rotation in the plane which is normal to the direction of propagation n, the parameters ξ_1 and ξ_3 change, while the sum $\xi_1^2 + \xi_3^2$ remains constant. The circular polarization doesn't change at such rotation. So, the quantities ξ_2 and $\xi_1^2 + \xi_3^2$ characterize a degree of circular and linear polarization and don't change not only at indicated rotations but at any Lorenz transformations.

Proceeding as at derivation of the spectral distribution Eq.(20) we find the polarization matrix:

$$d\varepsilon_{ik}(\mathbf{n},\omega) = e^2 \int\int [\mathbf{v}(t_1)-\mathbf{n}]_i[\mathbf{v}(t_2)-\mathbf{n}]_k e^{i[k(x(t_1)-x(t_2))]}dt_1dt_2\frac{d^3k}{(2\pi)^2}. \quad (29)$$

This polarization matrix can be presented in the standard form as

$$d\varepsilon_{ik}(\mathbf{n},\omega) = \frac{1}{2}d\varepsilon(\mathbf{n},\omega)\left[\delta_{ik} + \sum_n^3(\sigma_n)_{ik}\xi_n\right], \quad (30)$$

where the Stokes parameters ξ_n define the radiation polarization. If one introduce some external vector $\mathbf{e}$ and composes the matrix analogue of Eq.(23): $e_i^*e_k =$

$(1/2)(I+\boldsymbol{\sigma\eta})_{ik}), |\boldsymbol{\eta}|=1$, then $d\varepsilon_{ik}(\mathbf{n},\omega)$ can be projected onto this matrix:

$$d\varepsilon(\mathbf{n},\omega,\boldsymbol{\eta}) \equiv d\varepsilon_{ik}(\mathbf{n},\omega)e_i^*e_k = d\varepsilon(\mathbf{n},\omega)\frac{1}{4}\mathrm{Tr}[(I+\boldsymbol{\sigma\xi})(I+\boldsymbol{\sigma\eta})]$$

$$= d\varepsilon(\mathbf{n},\omega)\frac{1}{2}(1+\boldsymbol{\xi\eta}) = d\varepsilon(\mathbf{n},\omega)[1-|\boldsymbol{\xi}|+|\boldsymbol{\xi}|(1+\cos 2\varphi)] \quad (31)$$

Radiation from high energy particles

The radiation from ultrarelativistic particles ($\gamma \gg 1, \gamma = \varepsilon/m = 1/\sqrt{1-v^2}$) has a remarkable feature: the influence of the longitudinal component of the external field (with respect the particle velocity $\mathbf{v}$) is negligibly small comparing to the role of transverse component.

Indeed a force in relativistic mechanics is

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = \frac{m}{\sqrt{1-v^2}}\left[\dot{\mathbf{v}} + \frac{\mathbf{v}(\mathbf{v}\dot{\mathbf{v}})}{1-v^2}\right] \tag{32}$$

For the longitudinal component of the acceleration one has

$$\dot{v}_{\parallel} = \frac{(\dot{\mathbf{v}} \cdot \mathbf{v})}{v} = \frac{F_{\parallel}}{m\gamma^3}, \tag{33}$$

while for the transverse component one has

$$\dot{\mathbf{v}}_\perp = \frac{\mathbf{F}_\perp}{m\gamma}. \tag{34}$$

Substituting these expression into Eqs.(12),(15) respectively we see that for the same order of magnitude of the longitudinal and transverse force the contribution of the longitudinal component is negligibly small (of order of $1/\gamma^2$). Within this accuracy the radiation intensity is totally determined by the transverse force only. In this approximation $(\dot{\mathbf{v}} \cdot \mathbf{v}) = (1/2)d\mathbf{v}^2/dt = 0$ i.e. $\mathbf{v}^2$ is conserved.

The angular distribution of intensity (the energy radiated by a particle per unit tame t_0) follows from Eq.(7).

$$\frac{d\varepsilon(\mathbf{n})}{dt_0} = \frac{e^2}{4\pi}\frac{1}{(1-\mathbf{n}\cdot\mathbf{v})^5}\left[\dot{\mathbf{v}}_\perp^2(1-\mathbf{n}\cdot\mathbf{v})^2 - \frac{1}{\gamma^2}(\mathbf{n}\cdot\dot{\mathbf{v}}_\perp)^2\right]d\Omega \tag{35}$$

The characteristic combination is

$$1 - \mathbf{n}\cdot\mathbf{v} = 1 - v\cos\vartheta \simeq \frac{1}{2}\left(\frac{1}{\gamma^2} + \vartheta^2\right) + O\left(\frac{1}{\gamma^4}\right) \tag{36}$$

Substitution of Eq.(36) into Eq.(35) leads to the conclusion that the angular distribution of radiation from ultrarelativistic particle is a narrow, needle-like cone having angle $\vartheta \sim 1/\gamma$ with the axis directed along the particle's instantaneous velocity.

Characteristic cases of radiation

So the high-energy particles radiates along the particle velocity in narrow cone with angle $\vartheta \sim 1/\gamma$. Correspondingly interrelation between the total angle of particle deflection in external field and the angle $1/\gamma$ becomes essential. There are two characteristic cases.

1. The total angle of particle deflection is large comparing with $1/\gamma$. Then in the given direction $\mathbf{n}$ a particle radiates from the small fraction of the trajectory where the direction of the particle velocity is changed by the angle $\sim 1/\gamma$. This fraction is called the radiation formation length. Indeed the main contribution in the integral in Eq.(21) gives the region

$$\omega(t_1 - t_2) - \mathbf{k}(\mathbf{r}(t_1) - \mathbf{r}(t_2)) \sim 1 \qquad (37)$$

Taking into account that

$$\mathbf{k} = \sqrt{\mathcal{E}(\omega)}\omega\mathbf{n}, \quad \mathbf{r}(t) = \mathbf{v}t, \quad \mathcal{E}(\omega) = 1 - \frac{\omega_0^2}{\omega^2}, \quad \omega_0^2 = \frac{4\pi\alpha n_e}{m}, \quad \mathbf{nv} = v\cos\vartheta, \qquad (38)$$

where $\mathcal{E}(\omega)$ is the dielectric constant, ω_0 is the plasma frequency (in any medium $\omega_0 < 100$ eV), n_e is the electron density, we have from Eq.(37) for

$\vartheta \ll 1$

$$t_1 - t_2 = \Delta t \sim l_f(\omega) = \frac{2\varepsilon^2}{\omega m^2\left(1 + \gamma^2\vartheta^2 + \dfrac{\omega_p^2}{\omega^2}\right)}, \quad \omega_p = \omega_0\gamma. \tag{39}$$

If the Lorentz factor $\gamma = \varepsilon/m \gg 1$, then from Eq.(37) follows:

(a) ultrarelativistic particle radiates into the narrow cone with the vertex angle $\vartheta \leq 1/\gamma$ along the momentum of the initial particle, the contribution of larger angles is suppressed because of shortening of the formation length;

(b) the effect of the polarization of a medium described by the dielectric constant $\mathcal{E}(\omega)$ manifest itself for soft photons only when $\omega \leq \omega_b$, since on the Earth $\omega_0 < 100$ eV we have $\omega_b < \varepsilon\omega_0/m < 2 \cdot 10^{-4}\varepsilon$;

(c) when $\vartheta \leq 1/\gamma,\ \omega_0\gamma/\omega \ll 1$ the formation length is of the order

$$l_f \simeq l_{f0}(\omega) = \frac{2\varepsilon^2}{\omega m^2}. \tag{40}$$

All characteristics of radiation depend on instantaneous values of $\dot{\mathbf{v}}$ and $\mathbf{v}$ since these values do't varies on the radiation length. Such situation takes place i.e. in the magnetic bremsstrahlung.

2. The total deflection angle of particle in external field is lower or of the order $1/\gamma$. Then the whole radiation of the particle is concentrated inside the narrow cone with the opening angle $\sim 1/\gamma$. Naturally, in this case the characteristics of radiation are more sensitive to the shape of external field. This kind of motion is realized in undulators, for motion in the field of electromagnetic wave, for motion in single crystal's channel. Another important case is the bremsstrahlung in the Coulomb field.

Qualitative picture of radiation in the case I

Radiation occurs from the small part of the trajectory where the velocity of particle turns at an angle $\sim 1/\gamma$. The phase difference of waves emitted by the particle in the direction $\mathbf{n}$ at the times t_1 and t_2 is

$$\omega[(t_1 - t_2) - \mathbf{n}(\mathbf{r}(t_1) - \mathbf{r}(t_2))] \tag{41}$$

As soon as the phase difference becomes of the order of unity, the radiation from different points of trajectory becomes incoherent an breakdown of radiation into the direction $\mathbf{n}$ occurs. Let us take into account that the radiation take place from the small part of the trajectory where $|\Delta\dot{\mathbf{v}}| \simeq v/\gamma \simeq 1/\gamma$ and let us expand the dynamic characteristics entering in Eq.(41) over the time difference $\tau = t_2 - t_1$

$$\begin{aligned}
&\mathbf{r}(t_2) = \mathbf{r}(t_1) + \mathbf{v}(t_1)\tau + \dot{\mathbf{v}}(t_1)\tau^2/2! + \ddot{\mathbf{v}}(t_1)\tau^3/3! + \dots \\
&\Delta\varphi \simeq \omega\tau[1 - \mathbf{n}\cdot\mathbf{v} - \mathbf{n}\cdot\dot{\mathbf{v}}\tau/2 - \mathbf{n}\cdot\ddot{\mathbf{v}}\tau^2/6] \\
&\simeq \omega\tau[1 - \mathbf{n}\cdot\mathbf{v} - \mathbf{n}\cdot\dot{\mathbf{v}}\tau/2 + \dot{\mathbf{v}}^2\tau^2/6]. \qquad (42)
\end{aligned}$$

In the last line we take into account that the force acting on the particle can be considered as transverse one i.e $\mathbf{v}\dot{\mathbf{v}} = 0$. Then with the accuracy up to terms $\sim 1/\gamma^2$

$$\mathbf{n}\cdot\ddot{\mathbf{v}} = -\dot{\mathbf{v}}^2. \qquad (43)$$

In Eq.(42) the expression in square brackets has the order of magnitude $1/\gamma^2$. Indeed $1 - \mathbf{n}\cdot\mathbf{v} \simeq 1 - v \simeq 1/2\gamma^2$, radiation occurs from the small part of the trajectory $|\Delta\mathbf{v}| = \dot{\mathbf{v}}\tau \sim 1/\gamma$ $(\tau \sim 1/|\dot{\mathbf{v}}|\tau)$. Thus the phase difference

$$\Delta\varphi \sim \omega/|\dot{\mathbf{v}}|\gamma^3 \qquad (44)$$

The condition of radiation breakdown is $\Delta\varphi \sim 1$ determines the frequency upper boundary

$$\omega \sim \omega_c = |\dot{\mathbf{v}}|\gamma^3 \tag{45}$$

For higher frequencies the phase difference becomes of the order of unity so that inside the radiation cone there are waves of the given frequency traveling in the opposite phase. The spectral expansions of the radiation field include integrals of the kind $\int \exp(i\Delta\varphi)dt$. At high frequencies the integrand oscillates rapidly which results in mutual suppression. Such kind of integrals are exponentially small. So the radiation intensity drops exponentially at $\omega \gg \omega_c$. For frequencies $\omega \le \omega_c$ the phase difference $\Delta\varphi$ is small and the exponential term can be replaced by unity and the radiation behavior is determined by the pre-exponential factor. If this factor increases with frequency, the maximum of radiation intensity distribution over frequency is in the range where $\omega \sim \omega_c$. Thus, high-energy particles radiate mainly high harmonics compared to the motion characteristic frequencies, which are determined by the value $\dot{\mathbf{v}}$ (for example, for circular motion

$|\dot{\mathbf{v}}| \simeq 1/R \simeq \omega_L,\ \omega_L$ is the revolution frequency).

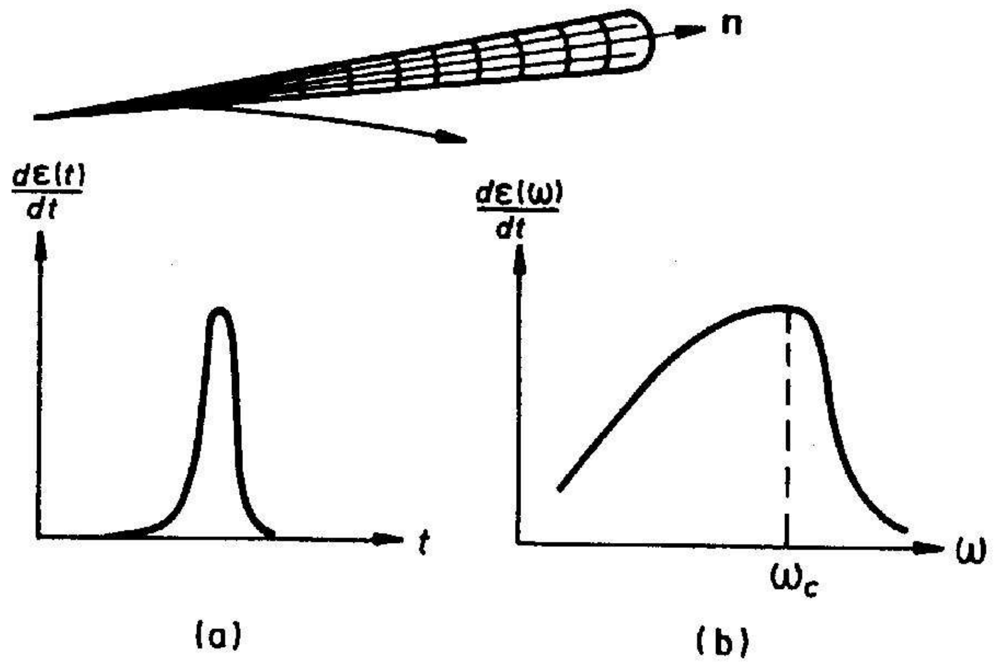

Figure 2: Time (a) and spectral distribution (b) of the radiation of ultrarelativistic particles.

Properties of radiation in magnetic bremsstrahlung limit

Here we consider the spectral distribution in form Eqs.(20) and(29). It is convenient to introduce the variables:

$$t_0 = \frac{t_1 + t_2}{2}, \quad \tau = t_2 - t_1, \quad t_1 = t_0 - \frac{\tau}{2}, \quad t_2 = t_0 + \frac{\tau}{2} \tag{46}$$

We consider the situation when the time of radiation in the given direction is much shorter that the time characteristics of particle motion. Having this fact in mind the particle kinematic characteristics can be expanded in a power series in τ:

$$\mathbf{v}_1 = \mathbf{v} - \dot{\mathbf{v}}\frac{\tau}{2} + \ddot{\mathbf{v}}\frac{\tau^2}{8}, \quad \mathbf{v}_2 = \mathbf{v} + \dot{\mathbf{v}}\frac{\tau}{2} + \ddot{\mathbf{v}}\frac{\tau^2}{8}, \quad \mathbf{r}_2 - \mathbf{r}_1 = \dot{\mathbf{v}}\tau + \ddot{\mathbf{v}}\frac{\tau^3}{24}, \tag{47}$$

where $\mathbf{v}_1 = \mathbf{v}(t_1)$ etc. Substituting this expansion in we find the following expression for the radiation intensity (energy emitted per unit particle time) of ultrarelativistic particles

$$dI(\mathbf{n},\omega) = \frac{d\varepsilon(\mathbf{n},\omega)}{dt_0} = e^2 \frac{d^3k}{(2\pi)^2} \int\limits_{-\infty}^{\infty} \left[2(1-\mathbf{n}\cdot\mathbf{v}) - \frac{1}{\gamma^2} - \frac{w^2\tau^2}{24}\right]$$
$$\times \exp\left[-i\omega\tau\left(1-\mathbf{n}\cdot\mathbf{v}+\frac{w^2\tau^2}{24}\right)\right] d\tau \tag{48}$$

Integrating by parts the term with $1-\mathbf{n}\cdot\mathbf{v}$ (analogous to the transition from

Eq.(20) to Eq.(21)) we obtain

$$dI(\mathbf{n},\omega) = -e^2 \frac{d^3k}{(2\pi)^2} \int_{-\infty}^{\infty} \left[\frac{1}{\gamma^2} + \frac{w^2\tau^2}{24}\right] \exp\left[-i\omega\tau\left(1 - \mathbf{n}\cdot\mathbf{v} + \frac{w^2\tau^2}{24}\right)\right] d\tau. \tag{49}$$

Here the integral over τ can be taken using the following integrals:

$$\int_{-\infty}^{\infty} \cos(bx + ax^3)dx = \frac{2}{3}\sqrt{\frac{b}{a}}K_{1/3}(\sigma), \quad \sigma = \frac{2}{3\sqrt{3}}\frac{b^{3/2}}{a^{1/2}},$$
$$\int_{-\infty}^{\infty} x\sin(bx + ax^3)dx = \frac{2}{3\sqrt{3}}\frac{b}{a}K_{2/3}(\sigma),$$
$$\int_{-\infty}^{\infty} x^2\cos(bx + ax^3)dx = -\frac{b}{3a}\int_{-\infty}^{\infty} \cos(bx + ax^3)dx. \quad (50)$$

Here $K_\nu(\sigma)$ is the Bessel function of the imaginary argument (MacDonald's function). Integrating Eq.(49) we obtain the spectral-angular distribution of radiation intensity

$$dI(\mathbf{n},\omega) = e^2\frac{d^3k}{(2\pi)^2}4\sqrt{\frac{2}{3}}\frac{\sqrt{1-\mathbf{n}\cdot\mathbf{v}}}{w}\left[4(1-\mathbf{n}\cdot\mathbf{v})-\frac{1}{\gamma^2}\right]K_{1/3}(\xi)$$
$$= \frac{e^2}{2\pi}\omega^2 d\omega x dx\frac{4}{\sqrt{3}}\frac{1}{\gamma^5}\frac{\sqrt{1+x^2}}{w}[2(1+x^2)-1]K_{1/3}(\xi), \qquad (51)$$

where in small angle approximation $1-\mathbf{n}\cdot\mathbf{v} = 1 - v\cos\vartheta \simeq 1 - v + \vartheta^2/2 \sim 1/2\gamma^2(+\gamma^2\vartheta^2) = (1+x^2)/2\gamma^2$,

$$\xi = \frac{4\sqrt{2}}{3}\frac{\omega}{w}(1-\mathbf{n}\cdot\mathbf{v})^{3/2} = \frac{2}{3}\frac{\omega}{w\gamma^3}(1+x^2)^{3/2} = \kappa(1+x^2)^{3/2}, \quad \kappa = \frac{2}{3}\frac{\omega}{w\gamma^3}. \qquad (52)$$

This expression gives the instantaneous properties of radiation when the velocity vector $\mathbf{v}(t_0)$ in the time moment t_0 is the bench mark of the reference system.

The argument of $K_{1/3}(\xi)$ in the radiation cone, i.e. when $1 - \mathbf{n}\cdot\mathbf{v} \sim 1/\gamma^2$, is of the order

$$\xi \sim \frac{\omega}{w\gamma^3} = \frac{\omega}{\omega_c} \tag{53}$$

The expansions of $K_\nu(z)$ at small and large values of argument are

$$K_\nu(z) \simeq \frac{\pi}{2z}e^{-z}, z \gg 1, \quad K_\nu(z) \simeq \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu, z \ll 1 \tag{54}$$

At $\xi \ll 1$, i.e. for frequencies $\omega \ll \omega_c$ for fixed angle inside radiation cone the spectral intensity has the form

$$dI(\vartheta,\omega) \simeq \frac{e^2}{\pi}\omega^{5/3}d\omega x dx \frac{\Gamma(1/3)}{3^{1/6}}\frac{\sqrt{1+x^2}}{w^{2/3}\gamma^4}[2(1+x^2)-1]. \tag{55}$$

For $\omega \gg \omega_c$ $(\xi \gg 1)$ we find

$$dI(\vartheta, \omega) \simeq \frac{e^2}{\sqrt{\pi}} \omega^{3/2} d\omega x dx \frac{2(1+x^2)-1}{w^{1/2}\gamma^{5/2}(1+x^2)^{1/4}} \exp\left[-\kappa(1+x^2)^{3/2}\right]. \qquad (56)$$

In this region the radiation intensity decreases exponentially.

The important characteristics of radiation is the spectral distribution integrated over radiation angle

$$dI(\mathbf{n}, \omega) = \frac{e^2}{2\pi} \frac{4}{\sqrt{3}} \frac{1}{\gamma^5} \omega^2 d\omega \int\limits_0^\infty x \frac{\sqrt{1+x^2}}{w} [2(1+x^2)-1] K_{1/3}(\xi) dx$$

$$= \frac{e^2}{2\pi} \frac{4}{\sqrt{3}} \frac{1}{\gamma^5} \omega^2 d\omega \int\limits_1^\infty \left(2s^{2/3}-1\right) K_{1/3}(\kappa s) \frac{ds}{3}, \qquad (57)$$

where the substitution was made $s = (1+x^2)^{3/2}$. Using the relation for K-function

$$\frac{d}{ds}[s^{2/3}K_{2/3}(\kappa s)] = -\kappa s^{2/3}K_{1/3}(\kappa s) \tag{58}$$

we obtain following expression for the spectral distribution of radiation intensity

$$dI(\omega) = \frac{e^2\omega d\omega}{\sqrt{3}\pi\gamma^2}\left[2K_{2/3}(\kappa) - \int_{\kappa}^{\infty} K_{1/3}(z)dz\right] \tag{59}$$

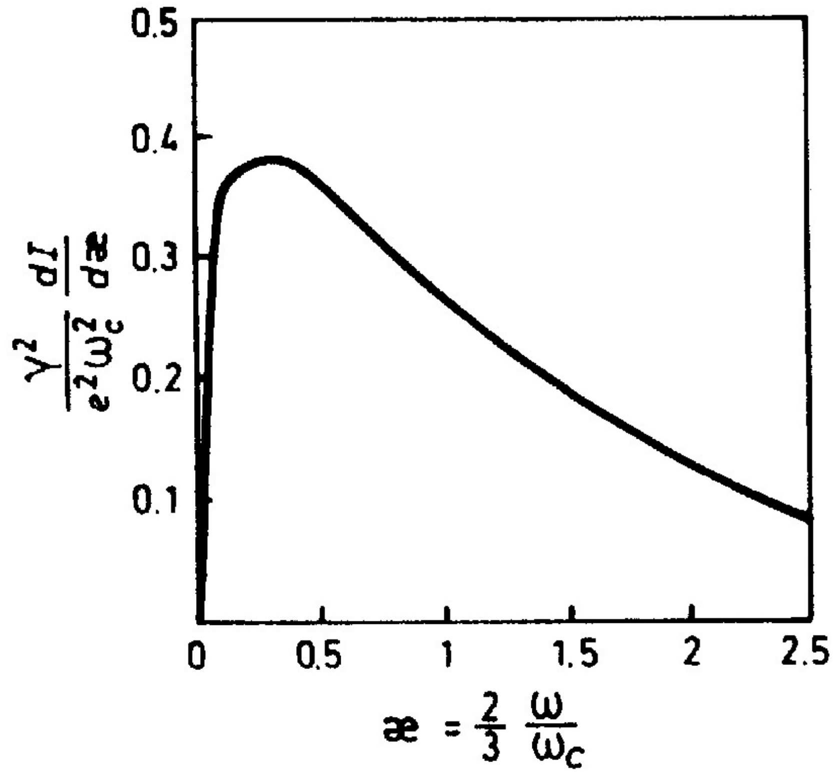

Figure 3: Dependence of the spectral intensity of radiation on the parameter κ

Quasiclassical method in high-energy QED

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Types of quantum effects in the radiation problem

Let us remind that in quantum theory dynamic variables become operators generally speaking non-commuting with each other.The simplest example is the commutator of the coordinate and momentum: $[x_i, p_k] = i\hbar\delta_{ik}$. The very important consequence is appearance of the uncertainty relation:

$$\Delta x_i \Delta p_k \geq \hbar\delta_{ik}. \tag{1}$$

In electromagnetic field the momentum operator is

$$P_\mu = p_\mu - eA_\mu = i\partial_\mu - eA_\mu \tag{2}$$

with commutation relation

$$[P_m, P_n] = ie\hbar\varepsilon_{mnk}H_k \tag{3}$$

and

$$[\mathbf{P}^2, P_n] = \Sigma_m[P_m^2, P_n] = 2ie\hbar\varepsilon_{mnk}H_kP_m, \tag{4}$$

where operator equality is used

$$[A^2, B] = A^2B - BA^2 = A^2B - ABA + ABA - BA^2 = A[A, B] + [A, B]A \quad (5)$$

We consider the quantum features in the motion with the relativistic Hamiltonian

$$\mathcal{H} = \sqrt{\mathbf{P}^2 + m^2}, \quad \mathbf{P} = -i\hbar\nabla - e\mathbf{A} \quad (6)$$

We introduce

$$\mathbf{b} \equiv [\mathcal{H}, \mathbf{P}], \quad \mathbf{c} \equiv [\mathcal{H}^2, \mathbf{P}] = [\mathbf{P}^2, \mathbf{P}] = 2ie\hbar(\mathbf{H} \times \mathbf{P}), \quad (7)$$

where $\mathbf{H}$ is the magnetic field, we used Eqs.(4),(5). Using equality Eq.(5) one has

$$\mathbf{c} = [\mathcal{H}^2, \mathbf{P}] = \mathcal{H}[\mathcal{H}, \mathbf{P}] + [\mathcal{H}, \mathbf{P}]\mathcal{H} = \mathcal{H}\mathbf{b} + \mathbf{b}\mathcal{H}, \quad (8)$$

or

$$\mathbf{c}\mathcal{H}^{-1} = \mathbf{b} + \mathcal{H}\mathbf{b}\mathcal{H}^{-1} = 2\mathbf{b} + [\mathcal{H}, \mathbf{b}]\mathcal{H}^{-1} \quad (9)$$

As a result we obtain the equation for $\mathbf{b}$

$$\mathbf{b} = \frac{1}{2}\mathbf{c}\mathcal{H}^{-1} - \frac{1}{2}[\mathcal{H}, \mathbf{b}]\mathcal{H}^{-1}, \tag{10}$$

which can be solved by successive iterations:

$$\mathbf{b} = \frac{1}{2}\mathbf{c}\mathcal{H}^{-1} - \frac{1}{2\cdot 2}[\mathcal{H}, \mathbf{c}]\mathcal{H}^{-2} + \ldots = ie\hbar(\mathbf{H}\times\mathbf{P})\mathcal{H}^{-1} - \frac{1}{2}(ie\hbar)^2(\mathbf{H}(\mathbf{HP}) - \mathbf{H}^2\mathbf{P})\mathcal{H}^{-3} - \ldots \tag{11}$$

It is seen that here the expansion is carried out in powers of

$$e\hbar < H\mathcal{H}^{-2} >= e\hbar\frac{H}{\varepsilon^2} = \frac{H}{\gamma^2 H_0} = \frac{\hbar\omega_0}{\varepsilon}, \tag{12}$$

where $\omega_0 = eH/\varepsilon$ is the Larmour frequency,

$$H_0 = \frac{m^2}{e\hbar} = \left(\frac{m^2c^3}{e\hbar}\right) = 4.41 \cdot 10^{13} Oe \tag{13}$$

is the critical magnetic field (for electron). The critical electric field (for electron) is

$$E_0 = \frac{m^2}{e\hbar} = \left(\frac{m^2 c^3}{e\hbar}\right) = 1.32 \cdot 10^{16} \frac{V}{cm} \tag{14}$$

There are two types of quantum effects at the radiation of high-energy particle in external field. The first one is associated with the quantization of the motion of the particle in the field just discussed above. One more example is the commutator of velocity components of the relativistic particles in a magnetic field **H** (where energy levels is $\varepsilon = \sqrt{m^2 + 2eH\hbar n} \gg m$) is

$$[v_i, v_k] = \frac{ie\hbar}{\varepsilon^2} \varepsilon_{ikj} H_j, \tag{15}$$

and the uncertainty relation for velocity components reads

$$\Delta v_i \Delta v_k \sim \frac{e\hbar H}{\varepsilon^2} = \frac{H}{H_0 \gamma^2} = \frac{\hbar\omega_0}{\varepsilon} \simeq \frac{1}{2n}, \tag{16}$$

so that with the energy rise the motion becomes increasingly classical.

The second type of quantum effect is associated with the recoil of the particle when it radiates and is of the order $\hbar\omega/\varepsilon$. Already in the classical limit ($\hbar\omega \ll \varepsilon$) this type is principal since $\omega \sim \omega_0 \gamma^3$

The Probability of Radiation in a Stationary External Field

The matrix element of the photon radiation by the charged particle in the external field in the first order of the perturbation theory with respect to the interaction with the radiation field follows from the power expansion of the S matrix and may, for particles with any spin, be represented in the form

$$U_{fi} = \frac{ie}{2\pi\sqrt{\hbar\omega}} \int dt \int d^3r F^+_{fs'}(\mathbf{r}) \exp(i\varepsilon_f t/\hbar)$$
$$(e^* J) \exp[i(\omega t - \mathbf{kr})] \exp(-i\varepsilon_i t/\hbar) F_{is}(\mathbf{r}), \qquad (17)$$

where $F_{is}(\mathbf{r})$ is the solution of the wave equation in the given field with the energy ε_i and in the spin state s, e^μ is the photon polarization vector, $k^\mu(\omega, \mathbf{k})$ is the photon 4-momentum, J^μ is the current vector.

For the states with large orbital momenta with which we are concerned the following approximation may be made:

$$\exp(-i\varepsilon_i t/\hbar)F_{is}(\mathbf{r}) = \Psi_s(\mathbf{P})\exp(-i\mathcal{H}t/\hbar)|i>, \quad P^\mu = i\hbar\partial^\mu - eA^\mu, \tag{18}$$

where $\Psi_s(\mathbf{P})$ is the operator form of the particle wave function in the spin state s in the given field. This form may be obtained from the free wave function via substitution of the variables by the operators: $\mathbf{p} \to \mathbf{P},\ \varepsilon \to \mathcal{H} = \sqrt{\mathbf{P}^2 + m^2}$. In the coordinate representation $|i>$ is the solution of the Klein-Gordon equation in the given field. Substituting Eq.(18) into Eq.(17) and taking into account that the Schrödinger operators, standing between the exponential factors $\exp(\pm i\mathcal{H}t/\hbar)$, convert into the explicitly time-dependent Heisenberg operators of the dynamic variables of the particle in the given field, we obtain the following formula for the matrix element Eq.(17):

$$U_{fi} = \frac{ie}{2\pi\sqrt{\hbar\omega}}\left\langle f|\int dt\exp(i(\omega t)M(t)|i\right\rangle, \tag{19}$$

where

$$M = \frac{1}{\sqrt{\mathcal{H}}}\Psi^+_{s'}(\mathbf{P})\{(e^*J), \exp[-i\mathbf{kr(t)}]\}\Psi_s(\mathbf{P})\frac{1}{\sqrt{\mathcal{H}}}; \tag{20}$$

here $\mathbf{P}(t), J^\mu(t), \mathbf{r}(t)$ are the Heisenberg operators of the particle momentum, current and coordinates respectively, the brackets $\{,\}$ denote the symmetrized product of operators (half of the anticommutator). Note, that $\Psi_s(\mathbf{P})\exp(-i\mathcal{H}t/\hbar)\ |i>$ is the operator solution of the wave equation.

It should be noted that in contrast to the case of free particles, where the matrix element of the first order becomes zero since the law of energy-momentum conservation cannot be fulfilled, here part of the recoil momentum takes up the field, in consequence of which $U_{if} \neq 0$.

For example for a particle with spin 0 (s) one has

$$M_s = \frac{1}{\sqrt{\mathcal{H}}}\{(e^* P), \exp[-i\mathbf{kr}(\mathbf{t})]\}\frac{1}{\sqrt{\mathcal{H}}} \tag{21}$$

For a particle with spin 1/2 (e) one has

$$M_e = \sqrt{\frac{m}{\mathcal{H}}}\bar{u_{s'}}(P)e^* \exp[-i\mathbf{kr}(\mathbf{t})]u_s(P)\sqrt{\frac{m}{\mathcal{H}}}, \tag{22}$$

where for the standard set of γ-matrix

$$\gamma^\mu \equiv (\gamma^0, \gamma^k), \quad \gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix} \tag{23}$$

(here I is the unit matrix of the rank two, σ^k are the Pauli matrices) one has

$$u_s(P) = \sqrt{\frac{\mathcal{H}+m}{2m}} \begin{pmatrix} \varphi[\boldsymbol{\zeta}(t)] \\ \frac{1}{\mathcal{H}+m}\boldsymbol{\sigma}\mathbf{P}\varphi[\boldsymbol{\zeta}(t)] \end{pmatrix}. \tag{24}$$

In this formula $\varphi[\boldsymbol{\zeta}(t)]$ is the two-component spinor describing the particle spin states in the moment of time t. There are two spinors $\varphi_1(\boldsymbol{\zeta})$ and $\varphi_2(\boldsymbol{\zeta})$ (spin up and down) so that

$$\varphi^+_{1,2}(\boldsymbol{\zeta})\boldsymbol{\sigma}\varphi_{1,2}(\boldsymbol{\zeta}) = \pm\boldsymbol{\zeta}, \quad \boldsymbol{\sigma}\boldsymbol{\zeta}\varphi_{1,2}(\boldsymbol{\zeta}) = \pm\varphi_{1,2}(\boldsymbol{\zeta}), \tag{25}$$

so that $\boldsymbol{\sigma}\boldsymbol{\zeta}$ is the polarization operator in the particle rest frame.

We are interested in the transition probability with photon radiation summed over the final particle states. Fulfilling this summation using the condition of the completeness of the vectors of the state $\sum_f |f><f| = 1$ we obtain for the transition probability with photon radiation:

$$dw = \frac{e^2}{\hbar}\frac{d^3k}{(2\pi)^2\omega}\left\langle i\left|\int dt_1 \int dt_2 \exp[i\omega(t_1 - t_2)]M^+(t_2)M(t_1)\right| i\right\rangle \tag{26}$$

Multiplying this formula by the energy of the radiated photon, we obtain the differential formula for the radiated energy:

$$d\varepsilon(\omega) = \hbar\omega dw(\omega) \tag{27}$$

Transformation of Operator Expressions

For further calculation, it is necessary to perform a series of actions with the operators appearing in the formula Eq.(27) We shall take out the operator $\exp[-i\mathbf{kr}(\mathbf{t_1})]$ in $M(t_1)$ on the left, and the operator $\exp[i\mathbf{kr}(\mathbf{t_2})]$ in $M^+(t_2)$ on the right, for which we use the relations

$$f(\mathbf{P})\exp(-i\mathbf{kr}) = \exp(-i\mathbf{kr})f(\mathbf{P} - \hbar\mathbf{k}),$$
$$\exp(i\mathbf{kr})f(\mathbf{P}) = f(\mathbf{P} - \hbar\mathbf{k})\exp(i\mathbf{kr}) \qquad (28)$$

where $f(\mathbf{P})$ is an arbitrary function, which is a consequence of the fact that $\exp(-i\mathbf{kr})$ is the displacement operator in momentum space, which may be verified using the commutation relations $[r_m, p_n] = i\hbar\delta_{mn}$ and expanding $\exp(-i\mathbf{kr})$ and $f(\mathbf{P})$ in a Taylor series. The variation of the function

$f(\mathbf{P})$ in Eq.(28) at commutation with $\exp(-i\mathbf{kr})$ (the "field" of the radiated photon) corresponds to making allowance for the recoil during radiation. After this operation has been performed, in each of the matrix elements only the commutative (with accuracy to the terms $\sim \hbar\omega/\varepsilon$ operators remain, and the subsequent problem consists in the consideration of the occurring combination $\exp[i\mathbf{kr}(\mathbf{t_2})]\exp[-i\mathbf{kr}(\mathbf{t_1})]$.

The Heisenberg operators $\mathbf{r}(t_2)$ and $\mathbf{r}(t_1)$, taken at various moments of time, do not commute with each other. This noncommutability is essential, so that to find the indicated combination one must not be limited by the expansion over the lower commutators. The calculation of the operator exponential formulae is generally referred to as "disentanglement". One of the basic propositions of the method described is the disentanglement of the operator formula $\exp[i\mathbf{kr}(\mathbf{t_2})]\exp[-i\mathbf{kr}(\mathbf{t_1})]$. We shall represent this formula in the form:

$$\exp[i\mathbf{kr_2}]\exp[-i\mathbf{kr_1}] = \exp(i\mathcal{H}\tau/\hbar)\exp[i\mathbf{kr_1}]\exp(-i\mathcal{H}\tau/\hbar)\exp[-i\mathbf{kr_1}], \quad (29)$$

where $\tau = t_2 - t_1$, here and below the indices 1 and 2 denote the dependence on the corresponding times. This representation is the consequence of the fact that $\exp(i\mathcal{H}\tau/\hbar)$ is the displacement operator in time. Taking account of the relation Eq.(28), we have:

$$\exp[i\mathbf{k}\mathbf{r_1}]\exp(-i\mathcal{H}(\mathbf{P_1})\tau/\hbar)\exp[-i\mathbf{k}\mathbf{r_1}] = \exp(-i\mathcal{H}(\mathbf{P_1} - \hbar\mathbf{k})\tau/\hbar), \tag{30}$$

Substituting this formula in Eq.(29) and introducing the notation

$$L_e(\tau) = \exp(-i\omega\tau)\exp[i\mathbf{k}\mathbf{r_2}]\exp[-i\mathbf{k}\mathbf{r_1}] = \exp(-ikx_2)\exp(ikx_1), \tag{31}$$

where $kx_1 = \omega t_1 - \mathbf{k}\mathbf{r_1}$, we obtain

$$L_e(\tau) = \exp(-i(\mathcal{H} - \hbar\omega)\tau/\hbar)\exp(-i\mathcal{H}(\mathbf{P_1} - \hbar\mathbf{k})\tau/\hbar) \tag{32}$$

To perform disentanglement in the combination Eq.(29), it is obviously sufficient to calculate $L_e(\tau)$. In order to obtain an explicit formula for it is necessary to differentiate Eq.(32) over τ:

$$\frac{L_e(\tau)}{d\tau} = \frac{i}{\hbar}\exp(i(\mathcal{H} - \hbar\omega)\tau/\hbar)[\mathcal{H} - \hbar\omega - \mathcal{H}(\mathbf{P_1} - \hbar\mathbf{k})]$$
$$\times\exp(-i(\mathcal{H} - \hbar\omega)\tau/\hbar) = \frac{i}{\hbar}[\mathcal{H} - \hbar\omega - \mathcal{H}(\mathbf{P_2} - \hbar\mathbf{k})]L_e(\tau) \qquad (33)$$

where we once again take advantage of the fact that $\exp(i\mathcal{H}\tau/\hbar)$ is the displacement operator in time. We shall take into account that

$$\mathcal{H}(\mathbf{P_2} - \hbar\mathbf{k}) = \sqrt{(\mathbf{P_2} - \hbar\mathbf{k})^2 + m^2} = \sqrt{(\mathcal{H} - \hbar\omega)^2 + 2\hbar k P_2 - (\hbar k)^2}, \qquad (34)$$

where

$$kP_2 = \omega\mathcal{H} - \mathbf{k}\mathbf{P}_2, \quad k^2 = \omega^2 - \mathbf{k}^2 \qquad (35)$$

For the real photons $k^2 = 0$, moreover, in accordance with (B.12)

$$kP_2 = \omega\mathcal{H} - \mathbf{k}\mathbf{P}_2 = \omega\mathcal{H}\left(1 - \frac{\mathbf{nP_2}}{\mathcal{H}}\right) = \omega\mathcal{H}\left[1 - \mathbf{nV_2} + O\left(\frac{\hbar\omega_0}{\varepsilon}\right)\right], \quad (36)$$

where $\mathbf{n} = \mathbf{k}/\omega$ is the unit vector in the direction of the photon's propagation. The equation Eq.(33) is analogous in form to the equation for the S matrix (the operators at $L_e(\tau)$ in the right part do not commute with each other at various moments of time). Therefore, the formal solution of Eq.(33) has the form

$$L_e(\tau) = T\exp\left[\frac{i}{\hbar}\int\limits_0^\tau \left(\mathcal{H} - \hbar\omega - \sqrt{(\mathbf{P}(t_1+\tau') - \hbar\mathbf{k})^2 + m^2}\right) d\tau'\right], \quad (37)$$

where T is the operator of chronological product, taking into account that $L_e(0) = 1$. (It should be noted that for the free particles the symbol T may be omitted since the momentum operators, taken at various moments of time, commute with each other.) The representation Eq.(34) and solution Eq.(37) are precise, the formula Eq.(36) is correct with accuracy to the terms $\sim \hbar\omega_0/\varepsilon$ (this is the accuracy of the accepted approximation). However, subsequently it is evidently necessary to take account of the characteristics of the radiation of ultrarelativistic particles, i.e. as in the classical radiation theory, to perform an expansion over the parameter $1/\gamma$. In the classical theory, the particle emits at an angle of $\sim 1/\gamma$, then $1 - \mathbf{nV} \sim 1/\gamma^2$. In the quantum theory, the mean values of the operators are of the type $1 - \mathbf{nV} \sim 1/\gamma^2$; in this sense it may be assumed that the operator $1 - \mathbf{nV}$ has the smallness $\sim 1/\gamma^2$. Taking this circumstance into account, one may perform the expansion

$$\mathcal{H}(\mathbf{P} - \hbar\mathbf{k}) = (\mathcal{H} - \hbar\omega)\left[1 + \frac{\hbar k P_2}{(\mathcal{H} - \hbar\omega)^2} - \frac{\hbar^2 k^2}{2(\mathcal{H} - \hbar\omega)^2} + ...\right] \qquad (38)$$

It is evident that this expansion is valid if the mean value $1 - \hbar\omega/\mathcal{H}$, i.e. the photon does not remove all the energy of the initial particle, so that the final particle is ultrarelativistic. Substituting the expansion Eq.(38) in the solution in the form Eq.(37), we see that the main terms in the exponent index cancel each other; in this way, for the real photons $k^2 = 0$

$$L_e(\tau) = T \exp\left[-\frac{i}{\hbar}\int\limits_0^\tau \frac{\mathcal{H}}{\mathcal{H}-\hbar\omega}\hbar kv(t_1+\tau')d\tau'\right], \tag{39}$$

where $kv = \omega(1-\mathbf{nv})$. In the integrand in Eq.(39) stands the value $kv \sim 1/\gamma^2$, and for calculation with this accuracy the noncommutability of the velocity operator at various moments of time may be neglected and the sign of the T product may be omitted, then

$$L_e(\tau) = \exp\left[\frac{\mathcal{H}}{\mathcal{H}-\hbar\omega}(kx_2 - kx_1)\right], \tag{40}$$

The derivation of the formula for $L_e(\tau)$ was not based on any particular characteristics of the external field, only the fact that the ultrarelativistic particle radiates into a cone with the angle $\sim 1/\gamma$ was used. Therefore, the result obtained is correct for the radiation of high-energy particles in the arbitrary external field, while the variation in values during commutation Eq.(28) corresponds to taking into account the energy and particle momentum variation (recoil) during radiation.

In this way, as a result of performing the disentanglement operation we have from Eqs.(31) and (40):

$$\begin{aligned} &\exp(-i\omega\tau)\exp[i\mathbf{k}\mathbf{r_2}]\exp[-i\mathbf{k}\mathbf{r_1}] \\ &\qquad = \exp(-ikx_2)\exp(ikx_1) \\ &\qquad = \exp\left[\frac{\mathcal{H}}{\mathcal{H}-\hbar\omega}(kx_2 - kx_1)\right], \end{aligned} \tag{41}$$

In view of the compensation of the main terms in the argument of the exponential in Eq.(40), the combination $\exp[i\mathbf{kr_2}]\exp[-i\mathbf{kr_1}]$ commutes with accuracy to the terms $\sim 1/\gamma^2$ with all the operators entering in $M^+(t_2)M(t_1)$. So, all the operators in the formula Eq.(26) prove to be commutative (with accuracy to the terms of the highest order over $\hbar\omega_0/\varepsilon$ and $1/\gamma^2$, and therefore after the disentanglement operation has been performed all those standing in the brackets corresponding to the initial state (the mean ones in the state with high quantum numbers), may be substituted for the corresponding classical values (c-numbers).

The Radiation Probability in Quantum Theory

Basing on performed derivation the equation for the transition probability Eq.(26) may be written in the form

$$dw = \frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\int dt_1 \int dt_2 R_2^* R_1 \exp\left[\frac{\varepsilon}{\varepsilon - \hbar\omega}(kx_2 - kx_1)\right], \qquad (42)$$

where $\alpha = \frac{e^2}{\hbar} = \left(\frac{e^2}{\hbar c}\right) = \frac{1}{137}$. For spin 1/2 particles

$$R = \sqrt{\frac{m}{\varepsilon'}}\bar{u}_{s'}(\mathbf{p}')\mathbf{e}\boldsymbol{\alpha}u_s(\mathbf{p})\sqrt{\frac{m}{\varepsilon}} = \varphi_{s'}^{+}O\varphi_s = \varphi_{s'}^{+}(A + i\boldsymbol{\sigma}\mathbf{B})\varphi_s, \qquad (43)$$

where α^k is the Dirac matrix

$$\alpha^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}. \tag{44}$$

In this expression $\varepsilon, \mathbf{p}, \varepsilon' = \sqrt{\mathbf{p}'^2 + m^2} \simeq \varepsilon - \hbar\omega, \mathbf{p}' = \mathbf{p} - \hbar\mathbf{k}$ are no longer operators but c-numbers, we have proceed to the two-component spinors.

The explicit expressions for A and $\mathbf{B}$ can be found starting from bi-spinor representation Eq.(24):

$$A = \frac{1}{2}\left(1 + \frac{\varepsilon}{\varepsilon'}\right)(\mathbf{ev})\mathbf{n} \simeq \frac{1}{2}\left(1 + \frac{\varepsilon}{\varepsilon'}\right)(\mathbf{e}\boldsymbol{\vartheta})$$

$$\mathbf{B} = \frac{\hbar\omega}{2(\varepsilon - \hbar\omega)}(\mathbf{e} \times \mathbf{b}, \quad \mathbf{b} = \mathbf{n} - \mathbf{v} + \frac{\mathbf{n}}{\gamma} \simeq -\boldsymbol{\vartheta} + \frac{\mathbf{n}}{\gamma} \tag{45}$$

where $\boldsymbol{\vartheta} = (1/v)(\mathbf{v} - \mathbf{n}(\mathbf{nv})) \simeq \mathbf{v}_\perp; \mathbf{v}_\perp$ is is the component of velocity transversal to the vector $\mathbf{n}$.

During the disentanglement operation the spin characteristics of the particles contained in the function $R(t)$ are not touched upon. This is linked to the fact that in our approximation the influence on the motion of the particle of the spin interaction with the external field (the terms $\sim \hbar\omega_0/\varepsilon$) may be disregarded. The function $R(t)$ describing the spin states has the form of a transition matrix element for free particles taking into account the law of momentum conservation.

Consequently, all the characteristics of radiation in the external field in Eq.(42) consist of the following: (1) the argument of the exponent contains the factor $\varepsilon/(\varepsilon - \hbar\omega)$ (taking into account the recoil, which is universal for all external fields); (2) the characteristics of the field appear in (2.28) in the dependence $\mathbf{p} = \mathbf{p}(t), kx = kx(t)$, since the evolution of the momentum and the coordinates in time are taken in the given field.

The formula for the quasiclassical matrix element Eq.(42) $R(t)\exp[i\varepsilon kx(t)/(\varepsilon-\hbar\omega)]$ is considerably simpler than the one obtained with the immediate integration of the exact solutions of the wave equation in and the fact that this formula is time-dependent via $\mathbf{p}=\mathbf{p}(t), kx=kx(t)$ in a defined sense is analogous to the classical theory, since it describes the radiation in terms of the trajectory. Therefore, in the given method, as in the classical theory of radiation, the interrelation between the total angle of deviation of the particle in the external field and the angle $1/\gamma$ proves to be significant. As a result of this we shall separately investigate two characteristic cases: (1) the total angle deviation is large in comparison to $1/\gamma$ (the case of motion in a macroscopic external field, for example in the magnetic field) and (2) the total angle of deviation of the particle in a field $\sim 1/\gamma$ (the case of radiation in quasiperiodic motion is an example of this situation).

In Eq.(42) enters the combination $R_2^*R_1=R^*(t_2)R(t_1)=R^*(t+\tau/2)R(t-\tau/2)$. If one uses the equation for classcial spin vector it may easily be verified that $\boldsymbol{\zeta}$ with accuracy to the terms $\sim 1/\gamma$ precesses with the same frequency as the velocity. In Case I the characteristic radiation time has the same order as in classical electrodynamics: $\tau \sim 1/w\gamma = 1/\omega_L\gamma$. Then

$$\varphi(\boldsymbol{\zeta}(t+\tau/2)) = \varphi(\boldsymbol{\zeta}(t)) + \frac{\tau}{2}\zeta_i\frac{\partial\varphi}{\partial\zeta_i} = \varphi(\boldsymbol{\zeta}(t)) + O\left(\frac{1}{\gamma}\right) \tag{46}$$

since in accordance with what has been said above $|\boldsymbol{\zeta}|\tau \simeq \dot{\mathbf{v}}\tau \sim 1/\gamma$. So, one can neglect the variation of spin vector during the radiation process and present entering in Eq.(42) the two-component spin density matrix in standard form $\sum_{s,s'}\varphi_s\varphi_{s'}^{+} = (1+\boldsymbol{\zeta}\boldsymbol{\sigma})/2$ and in accordance with what has been said above

$$\sum_{s,s'} R_2^* R_1 = \frac{1}{4}\mathrm{Tr}\left[(1+\boldsymbol{\zeta}_i\boldsymbol{\sigma})(A_2^* - i\boldsymbol{\sigma}\mathbf{B}_2^*)(1+\boldsymbol{\zeta}_f\boldsymbol{\sigma})(A_1 + i\boldsymbol{\sigma}\mathbf{B}_1)\right] \tag{47}$$

Spectral-angular distribution of radiation

Substituting Eq.(47) into Eq.(42) we can calculate any radiation characteristics, including polarization and spin ones. The intensity of the radiation is $dI = \hbar\omega dw$. After summation over spin states of the final electron and averaging over spin states of the initial electron the radiated energy has the form:

$$\frac{1}{2}\sum_{s_i,s_f} R_2^* R_1 = A_2^* A_1 + \mathbf{B}_2^* \mathbf{B}_1 \tag{48}$$

The amplitudes A and $\mathbf{B}$ depend on the photon polarization vector $\mathbf{e}$ (the Coulomb gauge is used here). The summation over photon polarization λ can performed using the formula

$$\sum_{\lambda} e_i e_k^* = \delta_{ik} - n_i n_k \tag{49}$$

After summation over the photon polarization one has

$$\sum_{\lambda} A_2^* A_1 = \frac{1}{4}\left(1 + \frac{\varepsilon}{\varepsilon'}\right)^2 (\mathbf{v}_1\mathbf{v}_2 - 1),$$

$$\sum_{\lambda} \mathbf{B}_2^* \mathbf{B}_1 = \frac{1}{4}\left(\frac{\hbar\omega}{\varepsilon'}\right)^2 \left(\mathbf{v}_1\mathbf{v}_2 - 1 + \frac{2}{\gamma^2}\right), \tag{50}$$

Here the terms of the higher order over $1/\gamma$ have been neglected and use has been made of the fact that terms of the type $\mathbf{nv}_1$ and $\mathbf{nv}_2$ may be represented in the form

$$\mathbf{n}\mathbf{v}_2 \exp\left(-i\frac{\varepsilon}{\varepsilon'}kx_2\right) = \left(\frac{i\varepsilon' d}{\omega\varepsilon dt_2} + 1\right)\exp\left(-i\frac{\varepsilon}{\varepsilon'}kx_2\right) \tag{51}$$

and the term with derivatives in the right part of Eq.(51) vanishes at integration over time within infinite limits. Performing the expansion of the terms entering Eq.(50) we obtain (the same expansion was made in classical radiation theory)

$$\mathbf{v}_{2,1} = \mathbf{v}\left(t \pm \frac{\tau}{2}\right) = \mathbf{v}(t) \pm \mathbf{w}\frac{\tau}{2} + \dot{\mathbf{w}}\frac{\tau^2}{8} + ...,$$
$$\mathbf{r}_{2,1} = \mathbf{r}\left(t \pm \frac{\tau}{2}\right) = \mathbf{r}(t) \pm \mathbf{v}\frac{\tau}{2} + \mathbf{w}\frac{\tau^2}{8} \pm \dot{\mathbf{w}}\frac{\tau^3}{48} + ... \tag{52}$$

taking into consideration that $\mathbf{v}\dot{\mathbf{v}} = O(1/\gamma^2)$ and $\mathbf{n}\dot{\mathbf{w}} = -\dot{\mathbf{v}}^2 + O(1/\gamma)$, we obtain

$$\mathbf{v}_1\mathbf{v}_2 = 1 - \frac{1}{\gamma^2} - \frac{w^2\tau^2}{2}$$

$$kx_2 - kx_1 = \omega\tau - \mathbf{kr}_2 + \mathbf{kr}_1 = \omega\tau\left(1 - \mathbf{nv} + \frac{w^2\tau^2}{24}\right) \tag{53}$$

Substituting Eqs.(48),(50) and (53) into Eq.(42) we have

$$dW_e(\mathbf{k}) = -\frac{\alpha}{(2\pi)^2}\frac{d^3k}{\hbar\omega}\int\limits_{-\infty}^{\infty}\left[\frac{1+u}{\gamma^2} + \left(1 + u + \frac{u^2}{2}\right)\frac{w^2\tau^2}{2}\right]$$

$$\times\exp\left[-i\frac{u\varepsilon\tau}{\hbar}\left(1 - \mathbf{nv} + \frac{w^2\tau^2}{24}\right)\right]d\tau, \tag{54}$$

where $u=\hbar\omega/\varepsilon'=\hbar\omega/(\varepsilon-\hbar\omega)$ is very important variable in quantum radiation theory. Integrating Eq.(54) with use of standard representation of MacDonald's functions, we obtain the spectral-angular distribution of radiation probability

$$dW_e(\mathbf{k})=\frac{\alpha}{\pi^2}\frac{d^3k}{\hbar\omega}\sqrt{\frac{2}{3}}\frac{\sqrt{1-\mathbf{nv}}}{w}\left\{-\frac{1+u}{\gamma^2}+2(1-\mathbf{nv})[1+(1+u)^2]\right\}K_{1/3}(\varsigma)$$

$$=\frac{\alpha}{\pi}\omega d\omega x dx\frac{2}{\sqrt{3}}\frac{1}{\gamma^3}\frac{\sqrt{1+x^2}}{\chi m}[(1+x^2)[1+(1+u)^2]-(1+u)]K_{1/3}(\varsigma), \qquad (55)$$

where

$$\varsigma=\frac{4\sqrt{2}}{3}\frac{u\varepsilon}{\hbar w}(1-\mathbf{nv})^{3/2}=\frac{2}{3}\frac{u}{\chi}[2\gamma^2(1-\mathbf{nv})]^{3/2}=\frac{2}{3}\frac{u}{\chi}[1+x^2]^{3/2}, \qquad (56)$$

in small angle approximation $1-\mathbf{nv}=1-v\cos\vartheta\simeq 1-v+\vartheta^2/2\sim 1/2\gamma^2(1+$

$\gamma^2\vartheta^2) = (1 + x^2)/2\gamma^2$. This expression gives the local properties of radiation when the particle's velocity vector is $\mathbf{v}$.

Here the very important parameter of quantum radiation theory appears

$$\chi = \frac{H}{H_0}\frac{p_\perp}{m} = \frac{\hbar}{m}|\dot{\mathbf{v}}_\perp|\gamma^2 = \frac{\hbar\omega_0}{\varepsilon}|\mathbf{v}_\perp|\gamma^3. \quad (57)$$

When $\chi \ll 1$ one has classical theory, when $\chi \geq 1$ we are in the quantum region.

Spectral distribution of radiation

The important characteristics of radiation is the spectral distribution integrated over radiation angle

$$dW(\omega) = \frac{\alpha}{\pi}\frac{2}{\sqrt{3}}\frac{1}{\gamma^2}\omega d\omega \int\limits_0^\infty x\frac{\sqrt{1+x^2}}{\chi\varepsilon}[[1+(1+u)^2](1+x^2)-(1+u)]K_{1/3}(\varsigma)dx$$

$$= \frac{2\alpha m^2}{3\sqrt{3}\pi\hbar\varepsilon}\frac{udu}{(1+u)^3}\int\limits_1^\infty \left\{[1+(1+u)^2]s^{2/3}-(1+u)\right\} K_{1/3}\left(\frac{2u}{3\chi}s\right) ds, \qquad (58)$$

where, as in classical theory, the substitution was made $s = (1+x^2)^{3/2}$.

Using the relation for K-function

$$\frac{d}{ds}[s^{2/3}K_{2/3}(\kappa s)] = -\kappa s^{2/3}K_{1/3}(\kappa s) \qquad (59)$$

we obtain following expression for the spectral distribution of radiation intensity

$$dI(\omega) = \frac{\alpha m^2}{\sqrt{3}\pi\hbar\epsilon}\frac{du}{(1+u)^3}\left\{[1+(1+u)^2]K_{2/3}\left(\frac{2u}{3\chi}\right) - (1+u)\int\limits_{2u/3\chi}^{\infty} K_{1/3}(y)dy\right\} \quad (60)$$

in the limit $u \ll 1$ this expression converts into the classical one.

The properties of radiation depend strongly on interrelation between $u = \hbar\omega/\varepsilon$ and χ. When $u \ll \chi$ one has

$$dI(u) = \frac{\alpha}{\pi}\frac{3^{1/6}\Gamma(2/3)}{2\hbar}\frac{\chi^{2/3}m^2u^{1/3}du}{(1+u)^4}[1+(1+u)^2], \quad (61)$$

and at $u \gg \chi$ one finds

$$dI(u) = \frac{\alpha m^2}{2\hbar} \left(\frac{\chi}{\pi}\right)^{1/3} \frac{u^{1/2} du}{(1+u)^4} (1 + u + u^2) \exp\left(-\frac{2u}{3\chi}\right). \tag{62}$$

In the range $u \ll 1$ or $\hbar\omega_0\gamma^3 \ll \varepsilon$ in all the essential regions $u \ll 1$ or $\hbar\omega \ll \varepsilon$ and $u/\chi = \omega/\omega_c$. The transition to this range coincides with transition $\hbar \to 0 (u \to \hbar\omega/\varepsilon, 1 + u \to 1)$. Then the obtained formulae convert into the classical ones. Consequently, at $\chi \ll 1$ the radiation picture is the same as in classical electrodynamics (with small quantum corrections) and in essential region $u \sim \chi$, the photon energy is much smaller than the energy of the radiating particle. As the energy increases, the value of $\hbar\omega_0\gamma^3$, increasing as γ^2, reaches the energy value ε (this is range $\chi = \hbar\omega_0\gamma^3/\varepsilon \geq 1$), and then the qualitative picture of the radiation becomes completely different from the classical one. At $\chi \sim 1$ in the essential region $\chi \sim u \sim 1, \hbar\omega \sim \varepsilon$.

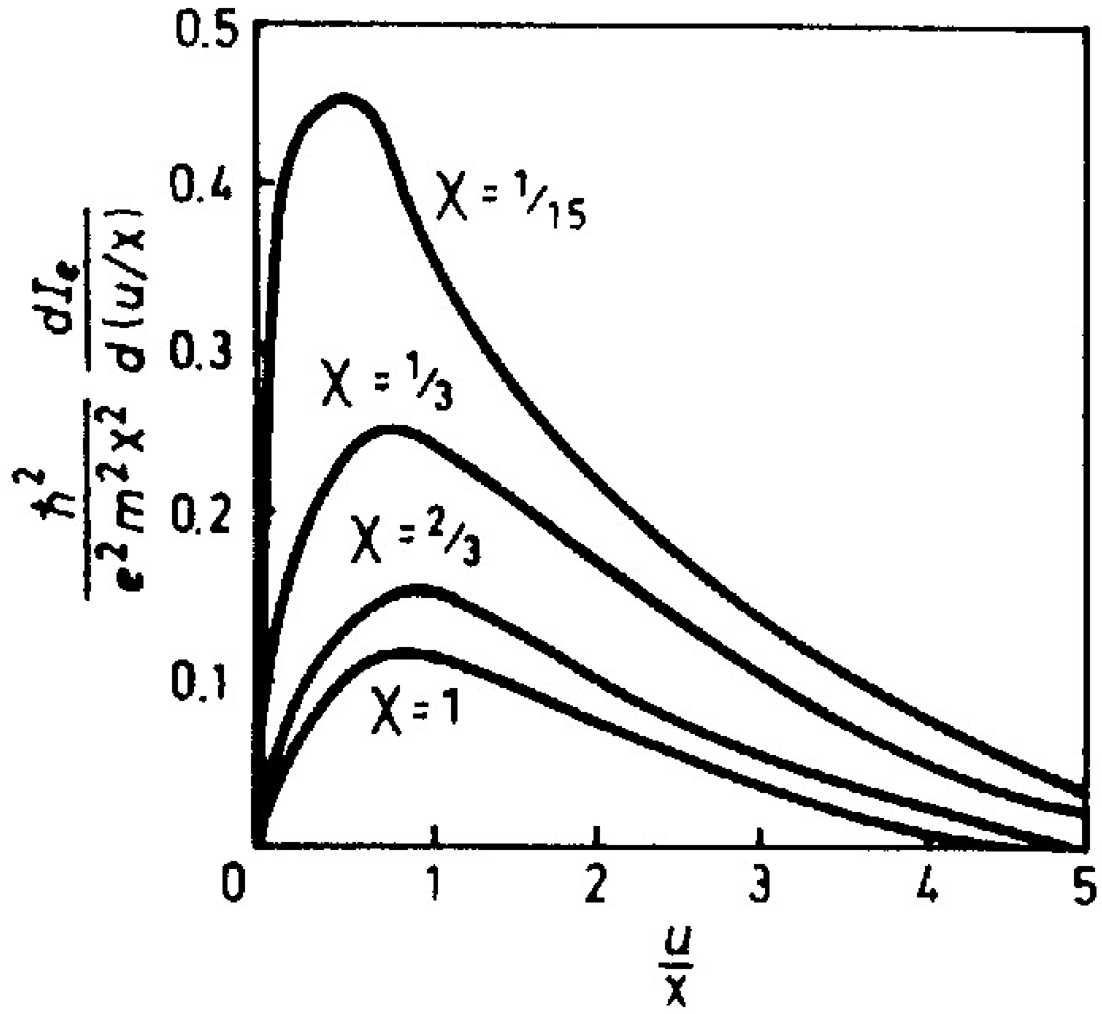

Figure 1: Dependence of the spectral intensity distribution on the parameter u/χ for indicated values of the parameter χ

We proceed now to the angular distribution of the radiation. . At $\chi \ll 1$ we have $\vartheta\gamma \sim 1$ or. $\vartheta \sim 1/\gamma$. This situation coincides with the classical one. This angular distribution is preserved at $\chi \sim 1$. But at $\chi \gg 1$ in the essential range $\vartheta\gamma \sim \chi^{1/3}$. This means that the angular distribution expands as $\vartheta \sim \chi^{1/3}/\gamma$.

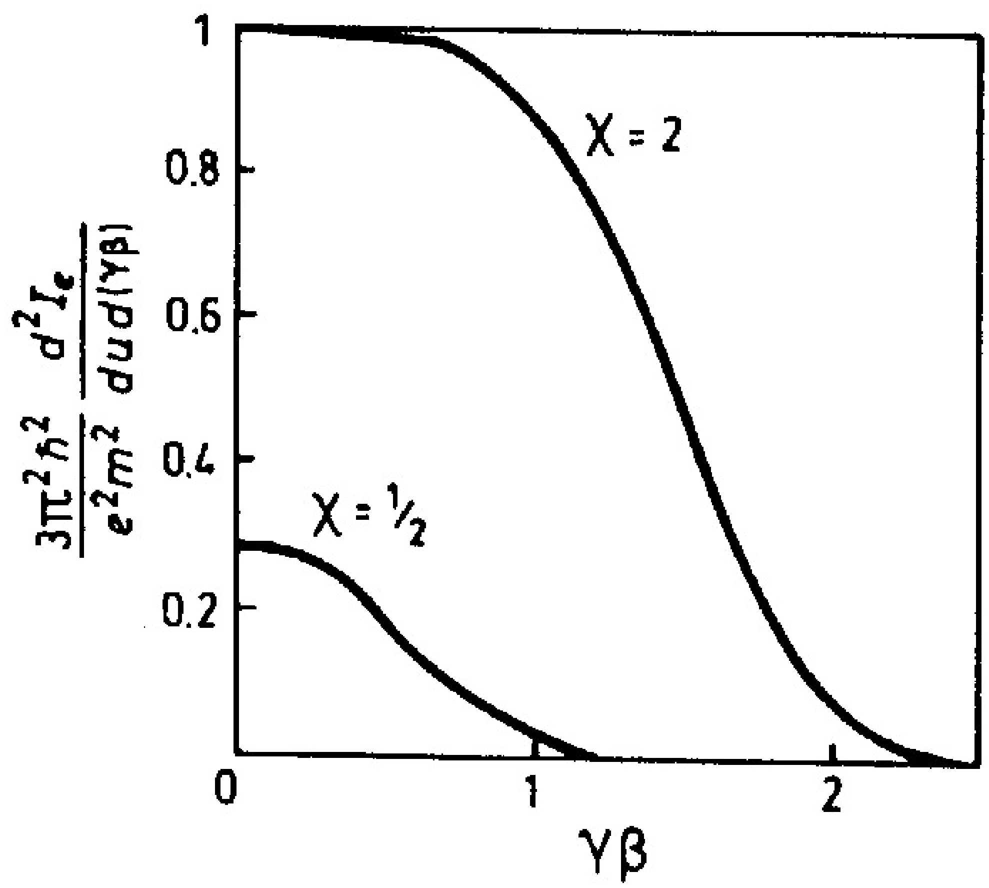

Figure 2: Dependence of the angular distribution on the polar angle times γ for indicated values of the parameter χ

Quasiclassical method in high-energy QED II

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Integral Characteristics of the Electron Radiation

Using the recurrent relation for K-functions

$$2\frac{dK_{2/3}(z)}{dz} + K_{1/3}(z) = -K_{5/3}(z), \tag{1}$$

one can obtained the following expression for the radiation intensity

$$dI(u) = \frac{\alpha m^2}{\sqrt{3}\pi\hbar^2}\frac{udu}{(1+u)^3}\left\{\frac{u^2}{1+u}K_{2/3}\left(\frac{2u}{3\chi}\right) + \int\limits_{2u/3\chi}^{\infty} K_{5/3}(y)dy\right\} \tag{2}$$

Using the obvious formula

$$\int\limits_0^{\infty} f(x)dx \int\limits_{x/\xi}^{\infty} \varphi(y)dy = \int\limits_0^{\infty} \varphi(y)dy \int\limits_0^{\xi y} f(x)dx, \tag{3}$$

we transform the integral term in Eq.(2)

$$\int\limits_0^\infty \frac{udu}{((1+u))^3}\int\limits_{u/\xi}^\infty K_{5/3}(y)dy = \int\limits_0^\infty K_{5/3}(y)dy\int\limits_0^{\xi y}\frac{udu}{((1+u))^3} = \int\limits_0^\infty K_{5/3}(y)\frac{(\xi y)^2}{((1+\xi y))^2}. \tag{4}$$

Using now the recurrent relation

$$z\frac{dK_{2/3}(z)}{dz} - \frac{2}{3}K_{2/3}(z) = -zK_{5/3}(z), \tag{5}$$

We will obtain the final expression for the total radiation intensity from electron in an external field

$$I_e(u) = \frac{e^2m^2}{3\sqrt{3}\pi\hbar^2}\int\limits_0^\infty \frac{u(4u^2+5u+4)}{(1+u)^4}K_{2/3}\left(\frac{2u}{3\chi}\right)du. \tag{6}$$

Using representation

$$\frac{1}{(1+u)^m} = \frac{1}{2\pi i} \int\limits_{\lambda - i\infty}^{\lambda + i\infty} \frac{\Gamma(-s)\Gamma(m+s)}{\Gamma(m)} u^s ds, \tag{7}$$

where $1 - m < \lambda < 0$. Then the integral over u can be easily taken using the following formula:

$$\int\limits_0^\infty x^\mu K_\nu(x) dx = 2^{\mu-1} \Gamma\left(\frac{1+\mu+\nu}{2}\right) \Gamma\left(\frac{1+\mu-\nu}{2}\right). \tag{8}$$

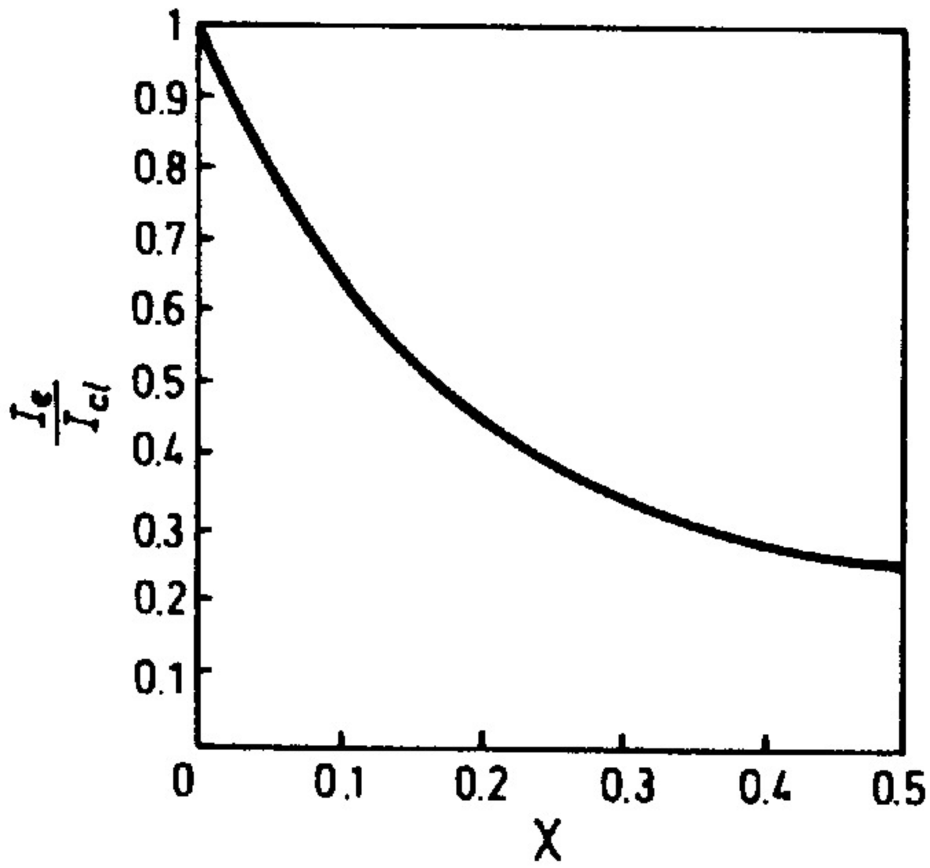

Figure 1: Ratio of quantum intensity of radiation to the classical one at low χ

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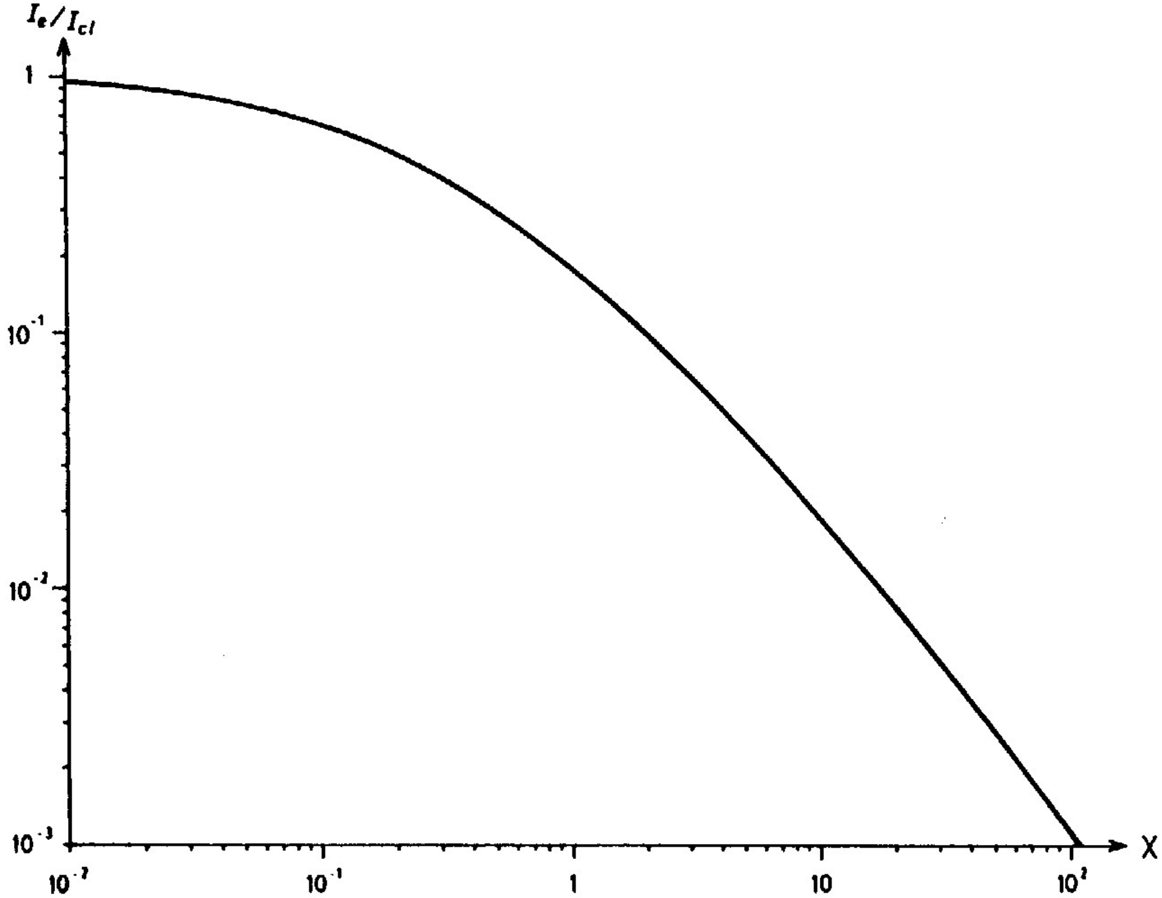

(b)

Figure 2: Ratio of quantum intensity of radiation to the classical one in a wide χ interval

Substituting Eq.(7) and than Eq.(2) into Eq.(6) we find the following expression for the total radiation intensity which is convenient for the asymptotic calculations

$$I_e(u) = \frac{e^2 m^2 \chi^2 \sqrt{3}}{8\pi\hbar^2 2\pi i} \int\limits_{\lambda - i\infty}^{\lambda + i\infty} (3\chi)^s (s^2 + 2s + 8)\Gamma(-s)\Gamma(s+2)\Gamma\left(\frac{s}{2} + \frac{2}{3}\right)\Gamma\left(\frac{s}{2} + \frac{4}{3}\right) ds \tag{9}$$

Asymptotic expansion

At $\chi \ll 1$ closing the integration contour on the right (the poles have $\Gamma(-s)$) we obtain the asymptotic series over power of χ

$$I_e(u) = \frac{e^2 m^2 \chi^2 \sqrt{3}}{8\pi\hbar^2} \sum_{k=0}^{\infty} (-1)^k (k+1)(k^2 + 2k + 8)\Gamma\left(\frac{k}{2} + \frac{2}{3}\right)\Gamma\left(\frac{k}{2} + \frac{4}{3}\right)(3\chi)^k$$

$$= \frac{2e^2 m^2 \chi^2}{3\hbar^2}\left[1 - \frac{55\sqrt{3}}{16}\chi + 48\chi^2 - \ldots\right] \tag{10}$$

The first term in this expression does not contain the Plank constant and represent the classical intensity ($I_{cl} = 2e^2 w^2 \gamma^4/3$). The second term IS THE FIRST QUANTUM CORRECTION (Sokolov, Klepikov, Ternov,1952; Schwinger, 1954) it is independent on spin of radiating particle. This dependence appears starting from χ^2 terms.

At $\chi \gg 1$ the integration contour should be closed on the left since the poles of the Γ functions lie at $s < 0$ and series over inverse powers of χ will be obtained

$$I_e(u) = \frac{32\Gamma(2/3)e^2m^2(3\chi)^{2/3}}{3^5\hbar^2}\left[1 - \frac{81}{16\Gamma(2/3)}(3\chi)^{-2/3} + \frac{165\Gamma(1/3)}{16\Gamma(2/3)}(3\chi)^{-4/3} - ...\right]$$

$$= 0.178\frac{e^2m^2}{\hbar^2}(3\chi)^{2/3}\left[1 - 3.74(3\chi)^{-2/3} + 20.4(3\chi)^{-4/3} - ...\right] \qquad (11)$$

The ratio I_e/I_{cl} is always smaller than 1. The series Eq.(10) is an asymptotic one, the numerical values of coefficient are increasing rapidly and at $\chi = 0.1$ one can't use it. One can see from Fig.1 that at $\chi = 0.1$ the quantum effects are essential and the intensity of radiation 1.5 times less than the classical intensity of radiation. Thus, the quantum effects in radiation are "turned on" rather early, when $\chi \ll 1$, while the region $\chi \sim 1$ is essentially quantum one.

Pair Creation by a Photon in an External Field

The Method of Investigation

The method developed for the investigation of the radiation process in an external field may easily be generalized for the investigation of other electromagnetic processes. In the lower order of the perturbation theory relating to the interaction with the radiation field, such processes are the production of a pair by a photon and single-photon annihilation of the pair in an external field. The matrix element of the pair production process has the form:

$$U_{fi} = \frac{ie}{(2\pi)^{3/2}\sqrt{\hbar\omega}} \left\langle q \right| \int dt \exp(-i(\omega t) M(t) \left| \bar{q} \right\rangle , \tag{12}$$

where

$$\begin{aligned} &\frac{1}{2}\exp(i\mathcal{H}t/\hbar)\frac{1}{\sqrt{\mathcal{H}}}\Psi_s^+(\mathbf{P},\mathcal{H})\{\hat{e}^*\exp[i\mathbf{kr}] \\ &+\exp[i\mathbf{kr}]\hat{e}\}\Psi_{\bar{s}}(\mathbf{P},-\mathcal{H})\frac{1}{\sqrt{\mathcal{H}}}\exp(i\mathcal{H}t/\hbar); \end{aligned} \tag{13}$$

here $\mathbf{P}(t), \mathbf{r}(t)$ are the Heisenberg operators of the particle momentum, current and coordinates respectively, the brackets $\{,\}$ denote the symmetrized product of operators (half of the anticommutator).

A particle (with positive frequency $\Psi_s(\mathbf{P}, \mathcal{H})$ and an antiparticle (with negative frequency $\Psi_{\bar{s}}(\mathbf{P}, -\mathcal{H}))$ are produced here in the electromagnetic vertex; $|q >, |\bar{q} >$ are state vectors describing the wave packets of the particle and the antiparticle; $s, \bar{s}$ are the spin state indices for particle and antiparticle.

Carrying the operator $\exp(i\mathcal{H}t/\hbar)$ in Eq.(13) on the right, which corresponds to the transition to Heisenberg operators

$$\exp(i\mathcal{H}t/\hbar)A = \exp(i\mathcal{H}t/\hbar)A\exp(-i\mathcal{H}t/\hbar)\exp(i\mathcal{H}t/\hbar) = A(t)\exp(i\mathcal{H}t/\hbar), \quad (14)$$

we obtain

$$M(t) = \frac{1}{\sqrt{\mathcal{H}}}\Psi_s^+(\mathbf{P}(t), \mathcal{H})\frac{1}{2}\{\hat{e}^* \exp[i\mathbf{kr}(\mathbf{t})]$$
$$+ \exp[i\mathbf{kr}(\mathbf{t})]\hat{e}\}\Psi_{\bar{s}}(\mathbf{P}(t), -\mathcal{H})\frac{1}{\sqrt{\mathcal{H}}}\exp(2i\mathcal{H}t/\hbar); \tag{15}$$

Carrying the operator $\exp[i\mathbf{kr}(\mathbf{t})]$ in Eq.(15) on the right using the relation of the type $f(\mathbf{P})\exp(-i\mathbf{kr}) = \exp(-i\mathbf{kr})f(\mathbf{P} - \hbar\mathbf{k})$, we have

$$M(t) = \frac{1}{\sqrt{\mathcal{H}}}\Psi_s^+(P)\frac{1}{2}\{\hat{e}^* + \hat{e}\}\Psi_{\bar{s}}(-P')\frac{1}{\sqrt{\mathcal{H}'}}\exp[i\mathbf{kr}(t)]\exp(2i\mathcal{H}t/\hbar), \tag{16}$$

where

$$\mathbf{P}' = \hbar\mathbf{k} - \mathbf{P}, \quad \mathcal{H}' = \mathcal{H}(\hbar\mathbf{k} - \mathbf{P}), \quad P = (\mathcal{H}(\mathbf{P}), \mathbf{P}),$$
$$P' = (\mathcal{H}(\hbar\mathbf{k} - \mathbf{P}), \hbar\mathbf{k} - \mathbf{P}) = (\mathcal{H}', \mathbf{P}'.) \tag{17}$$

We shall concern ourselves with the transition probability, summed over the final states of the pair produced. We shall perform the summation procedure in two stages. First, we shall sum over the final states of the antiparticle, using the condition of completeness $\sum |\bar{q} >< \bar{q}| = 1$, then we shall obtain

$$dw = \frac{\alpha}{(2\pi)^2\hbar\omega} \sum_q \left\langle q| \int dt_1 \int dt_2 \exp[-i\omega(t_2 - t_1)] M(t_2) M^+(t_1) |q \right\rangle \tag{18}$$

Disentanglement

We shall investigate the combination (similar to combination in radiation problem) entering this formula:

$$L_p(\tau) = \exp(-i\omega\tau) \exp[i\mathbf{k}\mathbf{r_2}] \exp(2i\mathcal{H}\tau/\hbar) \exp[-i\mathbf{k}\mathbf{r_1}], \tag{19}$$

which we shall represent in the form

$$L_p(\tau) = \exp(-i\omega\tau)\exp(i\mathcal{H}\tau/\hbar)\exp[i\mathbf{kr_1}]\exp(i\mathcal{H}\tau/\hbar)\exp[-i\mathbf{kr_1}]$$
$$= \exp(i(\mathcal{H} - \hbar\omega)\tau/\hbar)\exp(i\mathcal{H}(\mathbf{P_1} - \hbar\mathbf{k})\tau/\hbar) \qquad (20)$$

where we used the fact that $\exp(i\mathcal{H}t/\hbar)$ is the displacement operator over time while $\exp[i\mathbf{kr}]$ is the displacement operator in momentum space. The subsequent consideration is analogous to that performed in radiation problem. Having differentiated Eq.(20) over τ, we find

$$\frac{L_p(\tau)}{d\tau} = \frac{i}{\hbar}\exp(i(\mathcal{H} - \hbar\omega)\tau/\hbar)[\mathcal{H} - \hbar\omega + \mathcal{H}(\mathbf{P_1} - \hbar\mathbf{k})]$$
$$\times \exp(i(\mathcal{H} - \hbar\omega)\tau/\hbar) = \frac{i}{\hbar}[\mathcal{H} - \hbar\omega + \mathcal{H}(\mathbf{P_2} - \hbar\mathbf{k})]L_p(\tau). \qquad (21)$$

Integrating this equation and taking into account the initial condition $L_p(0) = 1$, we have

$$L_p(\tau) = T \exp\left\{ \frac{i}{\hbar} \left[(\mathcal{H} - \hbar\omega)\tau + \int\limits_{t_1}^{t_2} \sqrt{(\mathbf{P}(t) - \hbar\mathbf{k})^2 + m^2 dt} \right] \right\}, \qquad (22)$$

where T is the operator of chronological product.

Subsequently, it must be taken into account (as in radiation theory) that we shall investigate the case where the produced particle and antiparticle are ultrarelativistic. Below, we shall assert that at $H \ll H_0$, this is the case of main interest for physics. In this situation the main contribution to the probability of the process is made by the range of velocities of the final particle, for which $1 - \mathbf{nv} \sim 1/\gamma^2$, where $\mathbf{n}$ is the direction of motion of the photon. In physics terms, this means that the produced particle moves at an initial moment of time in the direction of the photon motion, while the photon-particle interaction remains essential, until the particle turns on an angle $\sim 1/\gamma$, so that the picture is very similar to that of magnetic bremsstrahlung, whereas the interaction on the length where particle turns at an angle $\sim 1/\gamma$ corresponds to the radiation from the coherence length. Taking into account that for real photons $k^2 = 0$, we have

(this result is very close to the radiation theory)

$$\mathcal{H}(\mathbf{P} - \hbar\mathbf{k}) = \sqrt{(\hbar\mathbf{k} - \mathbf{P})^2 + m^2} = \sqrt{(\mathcal{H} - \hbar\omega)^2 + 2kP}$$
$$= (\hbar\omega - \mathcal{H}) \left[1 + \frac{kP}{(\hbar\omega - \mathcal{H})} + \ldots\right] \tag{23}$$

On the basis of the arguments which led us to the final result in radiation theory, we have

$$L_p(\tau) = \exp\left[\frac{\mathcal{H}}{\hbar\omega - \mathcal{H}}(kx_2 - kx_1)\right], \tag{24}$$

The Probability of Pair Creation

Substituting this result in formula Eq.(18) and converting to the classical means (as in the radiation theory), we obtain the probability of pair production per all time of the interaction:

$$dw_p = \frac{\alpha}{(2\pi)^2}\frac{d^3p}{\hbar\omega}\int dt_1 \int dt_2 R_p(t_2) R_p^*(t_1) \exp\left[\frac{\varepsilon}{\varepsilon_f}(kx_2 - kx_1)\right], \qquad (25)$$

where

$$M = \frac{1}{2\sqrt{\varepsilon\varepsilon'}}\Psi_{s'}^{+}(p)[\hat{e}^* J(p) + \hat{e}J(-p')]\Psi_{\bar{s}}(-p');$$

$$\varepsilon_f = \hbar\omega - \varepsilon, \quad \varepsilon' = \sqrt{\mathbf{p}'^2 + m^2}, \quad \mathbf{p}' = \hbar\mathbf{k} - \mathbf{p} \qquad (26)$$

In Eq.(25) summation is performed over the final particle states taking into account that $\sum_q \to d^3p$. It is possible to arrive at Eqs.(24),(25) from formulae for radiation using the crossing symmetry: $p \to p'$ (substitution of notations for the outgoing particle); $p \to -p', k \to -k, s' \to s, s \to \bar{s}$ (substitution rule); $t_1 \leftrightarrow t_2$ (transition to uniform notation). At the substitution $p \to -p', \Psi_{\bar{s}}(p) \to \Psi_{\bar{s}}(-p')$ there is the transition from the incoming particle to the outgoing antiparticle, with such a substitution a wave with the positive frequency $\exp(ipx)$ transforms into a wave with negative frequency $\exp(-ip'x)$, so that $\Psi_{\bar{s}}(-p')$ describes the outgoing antiparticle.

For particles with spin $1/2$ the formula for $R_p(t)$ has the form (see Eqs.(16),(26))

$$R_p(t) = \frac{m}{\sqrt{\varepsilon\varepsilon_f}}\bar{u}_s(p)\hat{e}u_{\bar{s}}(-p') = \frac{m}{\sqrt{\varepsilon\varepsilon_f}}\bar{u}_s(p)\hat{e}v_{\bar{s}}(p') = \varphi_s^+ O\varphi_{\bar{s}} = \varphi_s^+(A(t) - i\boldsymbol{\sigma}\mathbf{B}(t))\varphi_{\bar{s}}, \tag{27}$$

where ε_f is defined in Eq.(26), bi-spinor $\bar{u}_s(p)$ describes outgoing electron, bi-spinor $v_{\bar{s}}(p')$ describes outgoing positron with corresponding 4-momentum. Here we passed to the two-component spinors φ_s. Within relativistic accuracy

$$A = \frac{1}{2}\left(1+\frac{\varepsilon}{\varepsilon_f}\right)(\mathbf{ev}) \simeq \frac{1}{2}\left(1+\frac{\varepsilon}{\varepsilon_f}\right)(\mathbf{e}\boldsymbol{\vartheta})$$

$$\mathbf{B} = \frac{\hbar\omega}{2\varepsilon_f}(\mathbf{e}\times\mathbf{b}), \quad \mathbf{b} = \mathbf{n}-\mathbf{v}+\frac{\mathbf{n}}{\gamma} \simeq -\boldsymbol{\vartheta}+\frac{\mathbf{n}}{\gamma}, \tag{28}$$

where $\boldsymbol{\vartheta} = (1/v)(\mathbf{v}-\mathbf{n}(\mathbf{nv})) \simeq \mathbf{v}_\perp; \mathbf{v}_\perp$ is is the component of velocity transversal to the vector $\mathbf{n}$.

As it was explained in the radiation theory, if one uses the equation for classical spin vector it may easily be verified that the vector $\boldsymbol{\zeta}$ with accuracy to the terms $\sim 1/\gamma$ precesses with the same frequency as the velocity. So the combination of the matrix elements entering in the expression for pair creation probability Eq.(25) can be written as

$$R_p(t_2)R_p^*(t_1) = R_p\left(t + \frac{\tau}{2}\right) R_p^*\left(t - \frac{\tau}{2}\right) =$$
$$\frac{1}{4}\mathrm{Tr}\left[(1+\boldsymbol{\zeta}_-\boldsymbol{\sigma})(A_2 - i\boldsymbol{\sigma}\mathbf{B}_2)(1+\boldsymbol{\zeta}_+\boldsymbol{\sigma})(A_1^* + i\boldsymbol{\sigma}\mathbf{B}_1^*)\right], \qquad (29)$$

where $(1+\boldsymbol{\zeta}\boldsymbol{\sigma})/2$ is the two-component spin density matrix, $A_1 = A(t_1)$ etc. The expression Eq.(25) with substitution Eq.(29) may be used for calculation of any characteristic of the process of pair production by a photon including polarization and spin characteristics. After summation over spin states of the produced electron and positron, we find from Eq.(29)

$$N_p = \sum_{s,\bar{s}} R_p(t_2)R_p^*(t_1) = 2(A_1^*A_2 + \mathbf{B}_1^*\mathbf{B}_2). \qquad (30)$$

After averaging over the photon polarization, we obtain

$$N_p = \frac{\varepsilon_f^2 + \varepsilon^2}{\varepsilon_f^2}(\mathbf{v}_1\mathbf{v}_2 - 1) + \frac{(\hbar\omega)^2}{2\varepsilon_f^2\gamma^2}. \tag{31}$$

The argument of the exponent in Eq.(25) can be written as

$$A \equiv \frac{\varepsilon}{\varepsilon_f}(kx_2 - kx_1) = \frac{\varepsilon\omega}{2\varepsilon_f}\int\limits_{t_1}^{t_2}\left[\frac{1}{\gamma^2} + (\mathbf{n} - \mathbf{v})^2\right]dt \tag{32}$$

where $\mathbf{n} = \mathbf{k}/\omega$. In the phase Eq.(31) at $\varepsilon_f = \hbar\omega - \varepsilon > 0$ there is compensation in γ_2 times, but at $\varepsilon_f < 0(\varepsilon > \hbar\omega)$ there is a sudden increase of the phase, which is leading to exponential suppression of the probability. In such a way, in this approach, the energy conservation law is realized.

Substituting Eqs.(29)-(31) and Eq.(32) into Eq.(25) we obtain explicit expressions for probability of production of the pair of particles with spin 1/2 in the external field of the general type in quasiclassical approximation

$$dw_p = \frac{\alpha m^2}{(2\pi)^2\hbar\omega}\frac{d^3p}{\varepsilon\varepsilon_f}\int dt_1 \int dt_2 \left\{1 - \frac{\varepsilon_f^2+\varepsilon^2}{\varepsilon_f^2}\gamma^2[\mathbf{v}(t_1)-\mathbf{v}(t_2)]^2\right\} e^{iA}, \quad (33)$$

where A is given by Eq.(32).

Performing the expansion of the terms entering Eq.(33) we obtain (the same expansion was made in classical radiation theory)

$$\mathbf{v}_{2,1} = \mathbf{v}\left(t \pm \frac{\tau}{2}\right) = \mathbf{v}(t) \pm \mathbf{w}\frac{\tau}{2} + \dot{\mathbf{w}}\frac{\tau^2}{8} + ..., \quad (34)$$

taking into consideration that $\mathbf{v}\dot{\mathbf{v}} = O(1/\gamma^2)$ and $\mathbf{n}\dot{\mathbf{w}} = -\dot{\mathbf{v}}^2 + O(1/\gamma)$, we obtain

$$\mathbf{v}_1\mathbf{v}_2 = 1 - \frac{1}{\gamma^2} - \frac{w^2\tau^2}{2}$$

$$kx_2 - kx_1 = \omega\tau - \mathbf{k}\mathbf{r}_2 + \mathbf{k}\mathbf{r}_1 = \omega\tau\left(1 - \mathbf{n}\mathbf{v} + \frac{w^2\tau^2}{24}\right) \qquad (35)$$

Differential probability of pair creation

Substituting Eqs.(34),(35) into Eq.(33) we have

$$dW_p(\mathbf{p}) \equiv \frac{dw_p}{dt} = \frac{\alpha}{(2\pi)^2}\frac{d^3p}{\hbar\omega}\frac{\varepsilon}{\varepsilon_f}\int\limits_{-\infty}^{\infty}\left[\frac{1}{\gamma^2} - \frac{(\varepsilon^2+\varepsilon_f^2)w^2\tau^2}{4\varepsilon\varepsilon_f}\right]$$

$$\times \exp\left[i\frac{\omega\varepsilon\tau}{\varepsilon_f}\left(1 - \mathbf{nv} + \frac{w^2\tau^2}{24}\right)\right]d\tau, \qquad (36)$$

The integral over τ may be taken with use of standard representation of MacDonald's functions

$$dW_p(\mathbf{p}) = \frac{\alpha}{\pi^2}\frac{d^3p}{\hbar\omega}\frac{\varepsilon}{\varepsilon_f}\sqrt{\frac{2}{3}}\frac{\sqrt{(1-\mathbf{nv})}}{w\gamma^2}\left\{1 + 2(1-\mathbf{nv})\frac{\varepsilon_f^2+\varepsilon^2}{\varepsilon\varepsilon_f}\gamma^2\right\}K_{1/3}(\varsigma), \quad (37)$$

where

$$\varsigma = \frac{2}{3}\frac{(\hbar\omega)^2}{\varepsilon\varepsilon_f\kappa}[2\gamma^2(1-\mathbf{nv})]^{3/2} \tag{38}$$

Here the very important parameter of the pair creation theory appears

$$\kappa = \frac{H}{H_0}\frac{\hbar k_\perp}{m}, \tag{39}$$

which is analogous to the parameter χ in the radiation theory.

The Integral Characteristics of Pair Creation

The Spectral Distribution

For the angular integration we shall use the angles ϑ and φ. The integration over the azimuthal angle is trivial, since integrand don't depend on φ. For integration over ϑ we change to the variable

$$z = z(\vartheta) = [2\gamma^2(1 - \mathbf{nv})]^{3/2}, \quad \mathbf{nv} = v\cos\vartheta \tag{40}$$

At lower limit $z(0) = [2\gamma^2(1 - v)]^{3/2} = 1 + O(1/\gamma^2)$, while $z(\pi) = 8\gamma^3(1 + O(1/\gamma^2))$. In view that at large z the integrand falls exponentially, the upper limit may be substituted for infinity. Than with accuracy to the terms $O(1/\gamma^2)$

we obtain the spectral distribution over the energy of one of the final particles:

$$dW_p(\omega, \varepsilon) = \frac{\alpha m^2}{\sqrt{3}\pi}\frac{d\varepsilon}{(\hbar\omega)^2}\left[\frac{\varepsilon_f^2 + \varepsilon^2}{\varepsilon\varepsilon_f}K_{2/3}(\xi) + \int\limits_{\xi}^{\infty} K_{1/3}(y)dy\right] \qquad (41)$$

where

$$\xi = \frac{2}{3}\frac{(\hbar\omega)^2}{\varepsilon\varepsilon_f\kappa} = \frac{2}{3}\frac{1}{x(1-x)\kappa} \qquad (42)$$

The form of the spectrum is presented in Fig.3. The spectrum is symmetric with respect to point $\varepsilon/\hbar\omega = x = 1/2$. It is seen that at small κ the spectral distribution is concentrated near $x \simeq 1/2$. If κ gets larger the distribution becomes wider and a plateau arises at all x except the edges $x \ll 1, (1-x) \ll 1$. If κ gets larger ($\kappa \gg 1$) a hollow arises in the center of the plateau a depth of which is increasing with κ, so that at $\kappa \gg 1$ the spectral distribution has peaks near the edges. At $\kappa \gg 1$ the main contribution in Eq.(41) gives the term without the integral. The analysis of this term shows that these peaks near the edges are situated at x or $(1-x) \sim 1.6/\kappa$, besides the height of distribution in the maximum does not depend on κ:$2A^{-1}dW_p/dx \simeq 0.33(A = \alpha mH/H_0)$, and the height of the distribution in the minimum at x = 1/2 is $2A^{-1}dW_p/dx \simeq 0.41\kappa^{-1/3}$ and it decreases when κ increases. The contribution of the peaks into the integral probability of the pair production accounts for $\kappa^{-2/3}$ of the total probability.

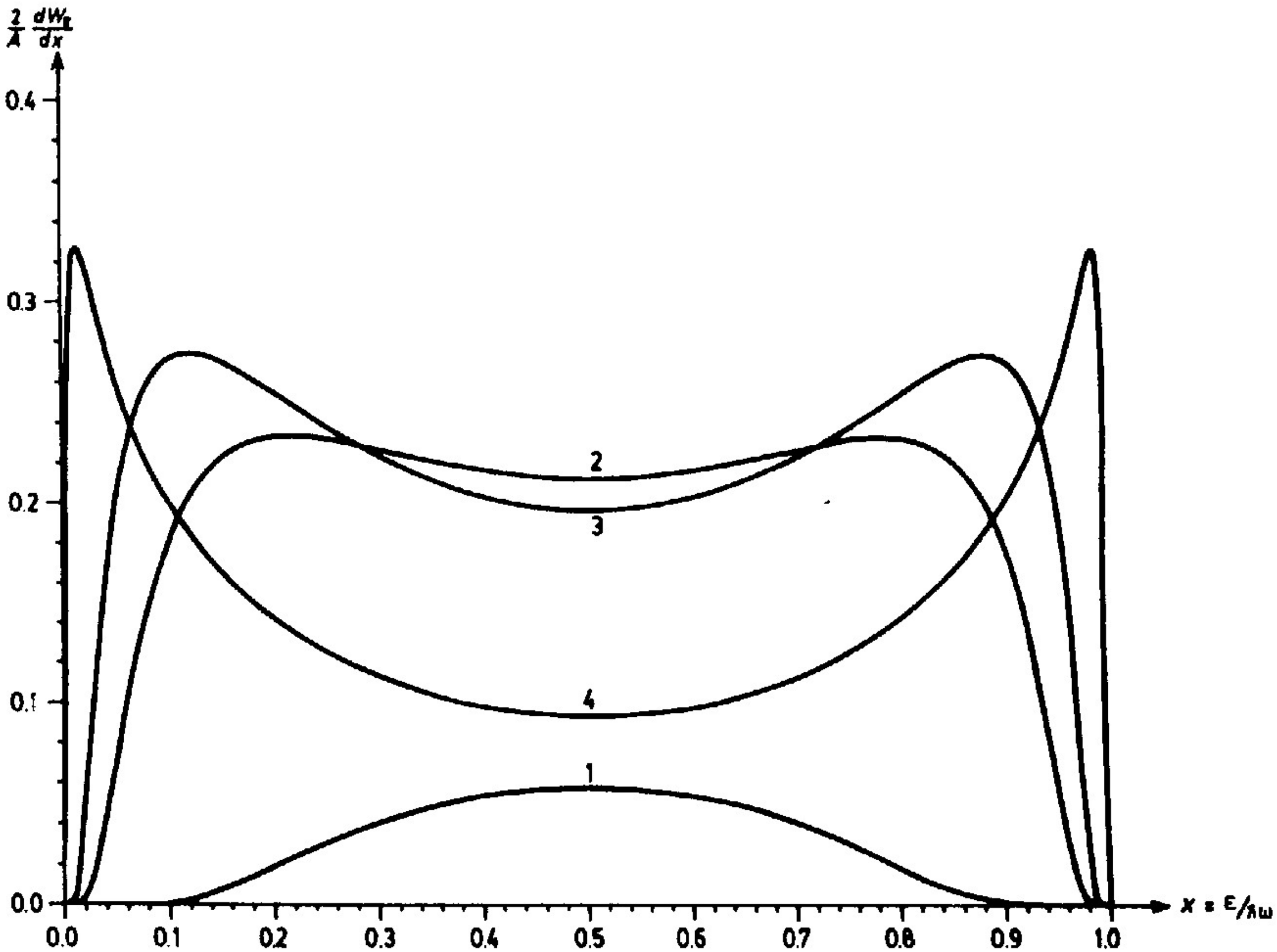

Figure 3: The spectrum of one of the particles of the created pair for different values of the parameter κ: $\kappa = 1$ curve 1; $\kappa = 5$ curve 2; $\kappa = 10$ curve 3;

At $\kappa \ll 1,\ \xi \gg 1$ and one can use the asymptotic expansion of MacDonald's function. Then from Eq.(41) we obtain asymptotic representation of the pair spectrum

$$\frac{dW_p}{dx} = \frac{A}{2}\sqrt{\frac{3\xi}{2\pi}}e^{-\xi}(1-x+x^2), \quad A = \frac{\alpha m H}{H_0} = \frac{\alpha m^2}{\hbar\omega}\kappa, \quad x = \frac{\varepsilon}{\hbar\omega}. \tag{43}$$

Even though this expression is derived at $\kappa \ll 1$, it gives the pair spectrum at $\kappa = 1$ with an accuracy better than 3%.

At $\kappa \gg 1$ and in the interval of x :$x \gg 1/\kappa$ or $(1-x) \gg 1/\kappa$ we have $\xi \ll 1$ and one can use an asymptotic expansion of MacDonald's function. The main contribution to probability Eq.(41) gives the term without integral. As a result we obtain

$$\frac{dW_p}{dx} = \frac{A 3^{1/6}\Gamma(2/3)}{2\pi\kappa^{1/3}}\frac{1}{[x(1-x)]^{1/3}}(x^2+(1-x)^2). \tag{44}$$

The terms $\sim \kappa^{-2/3}$ are dropped. The distribution Eq.(44) describes the spectrum in all the regions beyond the peaks near the edges.

The Integral Probability

To fulfill the integration over the spectrum, it is necessary in the integrals over y containing $\int\limits_{\xi}^{\infty} K_{1/3}(y)dy$ to perform integration by parts as it was done in calculation of the total intensity of radiation. As a result one obtains

$$W_p = \frac{\alpha m^2}{3\sqrt{3}\pi\hbar\omega}\int\limits_0^{\infty}\frac{9-v^2}{1-v^2}K_{2/3}(\eta)dv = \frac{\alpha m^2}{6\sqrt{3}\pi\hbar\omega}\int\limits_1^{\infty}\frac{8u+1}{u^{3/2}\sqrt{u-1}}K_{2/3}(\eta)du, \tag{45}$$

where

$$u = \frac{1}{(1-v^2)}, \quad \eta = \frac{8}{3\kappa(1-v^2)} = \frac{8u}{3\kappa} \tag{46}$$

We consider the dependence of the total probability on values of the parameter κ. At $\kappa \ll 1$ and at any $x, \xi \gg 1$ therefore in all the range of variations x use the asymptotic expansion of MacDonald's function. Then the integrals in Eq.(45) may easily be calculated. As a result, we find the series over the powers of κ the main terms of which is

$$W_p = \frac{3\sqrt{3}\alpha m^2 \kappa}{16\sqrt{2}\hbar\omega} \exp\left(-\frac{8}{3\kappa}\right) \left[1 - \frac{11}{16}\kappa + \frac{7585}{73728}\kappa^2 + ...\right] \tag{47}$$

Although the expression Eq.(47) was obtained under the assumption that $\kappa \ll 1$, comparing the results of numerical calculations of Eq.(47) and exact formula Eq.(45) one can be convinced that Eq.(47) represents the probability with an accuracy of better 15% up to $\kappa = 2$.

In the range $\kappa \ll 1$ the probability of the process is exponentially small. This situation is characteristic for all processes with a finite discontinuity of the square of the four momentum of the system (invariant mass of the system). For the pair production process, the discontinuity of the four momentum of the system is $\Delta_p^2 = (p+p')^2 - k^2 = (p+p')^2,\ \Delta_{pmin}^2 = 4m^2$.

At $\kappa \gg 1$ the asymptotic expansion of pair creation probability can be calculated using the asymptotic expansion of MacDonald's function. It is

$$W_p = C\frac{\alpha m^2 \kappa^{2/3}}{\hbar\omega} = CA\kappa^{-1/3}, \quad C = \frac{5\Gamma(5/6)(2/3)^{1/3}}{14\Gamma(7/6)} = 0.37961.. \qquad (48)$$

The function W_p/A is given in Fig.4. It reaches the maximum 0.11 at $\kappa \simeq 11.7$.

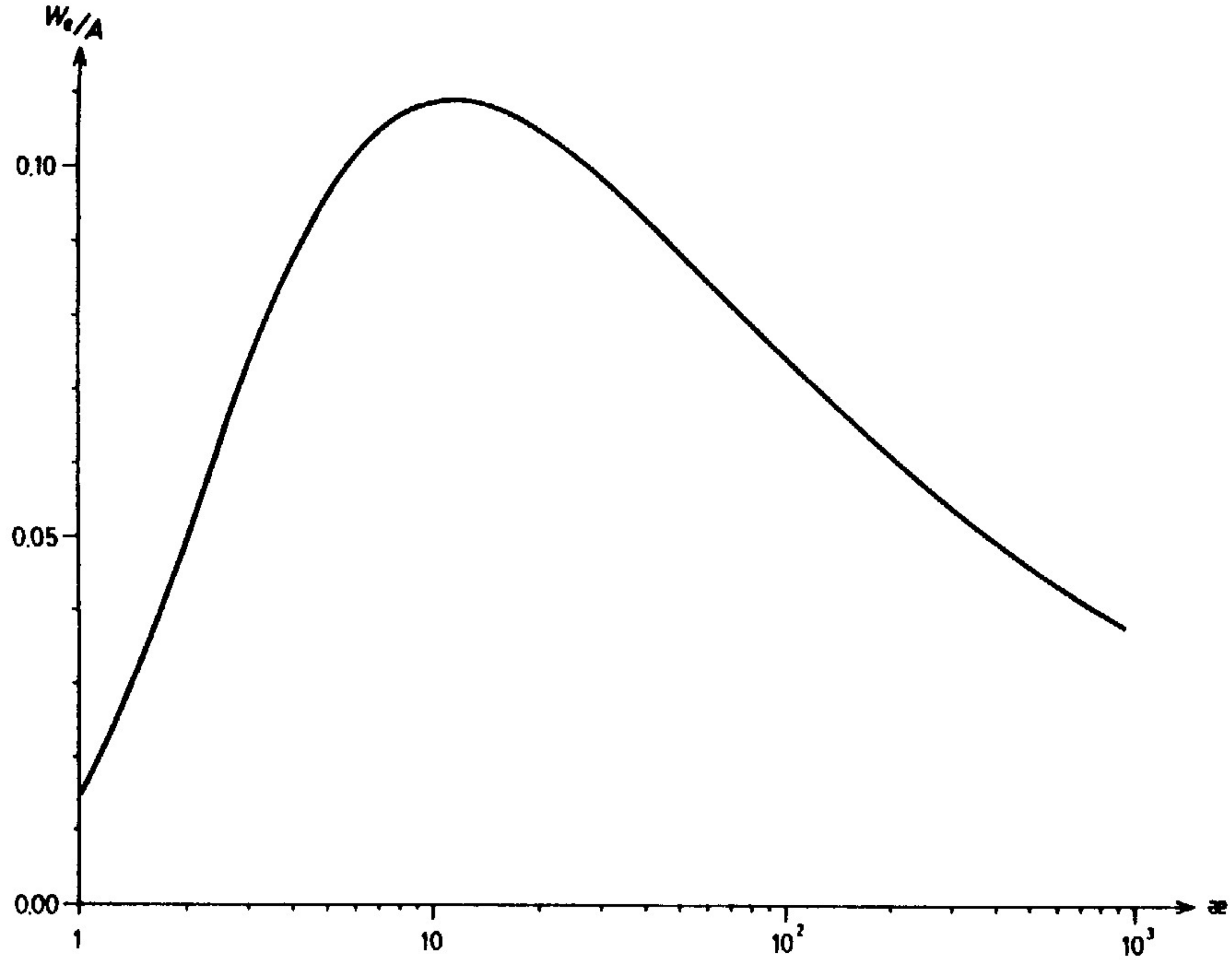

Figure 4: Probability of pair creation as a function of κ

Radiation and Pair Creation in Oriented Crystal

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The Crystal Potential

Description of crystal

An ideal crystal is built up by the repetition of a structural motif - some particular atomic group by translation in three dimensions. There are only fourteen distinct lattices (Bravais lattices). They me be determined by a set of space-lattice vectors

$$\mathbf{l} = l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + l_3\mathbf{a}_3 \tag{1}$$

where $l_1, l_2, l_3 = 0, \pm 1, \pm 2, \ldots$ The unit cell of atomic structure may be enclosed by the parallelepiped generated by the different translations corresponding to base vectors $\mathbf{a}, \mathbf{a}_2, \mathbf{a}_3$. If there is s atoms with coordinate $\mathbf{r}_a$ in the unit cell, the distribution of atoms in a crystal may be represented by the function

$$\varrho(\mathbf{r}) = \sum_{\mathbf{l}} \sum_{a=1}^{s} \delta(\mathbf{r} - \mathbf{r}_a - \mathbf{l}) \tag{2}$$

The distribution of atoms in a crystal is described by the periodic function

$$\varrho(\mathbf{r}) = \varrho(\mathbf{r} + \mathbf{l}) \tag{3}$$

The crystal potential is also periodic

$$U(\mathbf{r}) = U(\mathbf{r} + \mathbf{l}) \tag{4}$$

The periodic function $U(\mathbf{r})$ can be represented as Fourier series

$$U(\mathbf{r}) = \sum_{\mathbf{q}} G(\mathbf{q}) e^{-i\mathbf{q}\mathbf{r}}, \tag{5}$$

where the vectors $\mathbf{q}$, over which the summation is carried out, satisfy the condition $e^{-i\mathbf{q}\mathbf{l}} = 1$ according with Eq.(4).Than

$$\mathbf{q}\mathbf{l} = 2\pi N, \quad N \text{ is an integer} \tag{6}$$

It follows from Eq.(6) that the vector $\mathbf{q}$ can be represented in form

$$\mathbf{q} = 2\pi(n_1\mathbf{b}_1 + n_2\mathbf{b}_2 + n_3\mathbf{b}_3) \tag{7}$$

where n_1, n_2, n_3 are integers, and the vectors $\mathbf{b}_j$, called the reciprocal lattice vectors, are chosen to satisfy the relation

$$\mathbf{b}_j\mathbf{a}_i = \delta_{ji} \tag{8}$$

The explicit form of $\mathbf{b}_j$ is

$$\mathbf{b}_1 = (\mathbf{a}_2 \times \mathbf{a}_3)\Omega_0^{-1}, \quad \mathbf{b}_2 = (\mathbf{a}_3 \times \mathbf{a}_1)\Omega_0^{-1}, \quad \mathbf{b}_3 = (\mathbf{a}_1 \times \mathbf{a}_2)\Omega_0^{-1}, \tag{9}$$

where $\Omega_0 = (\mathbf{a}_1(\mathbf{a}_2 \times \mathbf{a}_3))$ is the volume of the unit cell. For example, the reciprocal lattice for primitive cubic lattice with edge length a will be the lattice of the same type with edge length $b = 1/a,$. If we multiply the vector $\mathbf{q}$ Eq.(7) by arbitrary direct lattice vector $\mathbf{r} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3$, than the equation

$$\mathbf{qr} = 2\pi(n_1m_1 + n_2m_2 + n_3m_3) = 2\pi N \tag{10}$$

determines a family of parallel crystal planes. If the integers n_1, n_2, n_3 are relatively prime, they are the Miller indices of these planes.

According to Eq.(3), the density $\varrho(\mathbf{r})$ can be represented as the Fourier series

$$\varrho(\mathbf{r}) = \Omega_0^{-1} \sum_{\mathbf{q}} S(\mathbf{q}) e^{-i\mathbf{q}\mathbf{r}}, \tag{11}$$

where $\Omega_0 = l^3$ for a cubic lattice, l is the lattice constant (length of the cubic edge) and the Fourier component $S(\mathbf{q})$ is determined by the formula

$$S(\mathbf{q}) = \int \varrho(\mathbf{r}) e^{i\mathbf{q}\mathbf{r}} d^3 r \tag{12}$$

Integration in the last expression is carried out over one unit cell. If we represent the density $\varrho(\mathbf{r})$ in the form $\varrho(\mathbf{r}) = \sum_{a=1}^{s} \delta(\mathbf{r} - \mathbf{r}_a)$, (see Eq.(2)) we obtain

$$S(\mathbf{q}) = \sum_{a=1}^{s} e^{i\mathbf{q}\mathbf{r}_a} \tag{13}$$

The quantity is called a geometric structure factor or simply **structure factor**. It appears in the description of the X-ray diffraction, pair production and photon emission process in a crystal.

Thermal vibrations

Let us present the potential of an individual atom of the crystal lattice as the Fourier integral

$$\varphi_a(\mathbf{r}) = \frac{1}{(2\pi)^3} \int g_a(\mathbf{q}) e^{-i\mathbf{qr}} d^3q \tag{14}$$

Atoms deviate from their equilibrium locations in crystal lattice owing to thermal(zero) vibrations. The probability $w(\mathbf{x})$ of a displacement $\mathbf{x}$ has the Gaussian form

$$w(\mathbf{x}) = \frac{1}{(2\pi u_1^2)^{3/2}} \exp\left(-\frac{x^2}{2u_1^2}\right), \quad \int w(\mathbf{x}) d^3x = 1, \tag{15}$$

where u_1 is the amplitude of thermal vibrations. The potential of an individual atom, averaged over thermal displacements, has form

$$\varphi(\mathbf{r}) = \int \varphi_a(\mathbf{r}-\mathbf{x})w(\mathbf{x})d^3x \tag{16}$$

Using Eqs.(14),(15) we find for the Fourier transform of the potential Eq.(16)

$$g(\mathbf{q}) = \int \varphi(\mathbf{r})e^{i\mathbf{qr}}d^3x = \int \varphi_a(\mathbf{r}-\mathbf{x})w(\mathbf{x})e^{i\mathbf{qr}}d^3x$$
$$= \int \varphi_a(\mathbf{r})e^{i\mathbf{qr}}d^3r \int w(\mathbf{x})e^{i\mathbf{qx}}d^3x = g_a(\mathbf{q})\exp\left(-\frac{u_1^2\mathbf{q}^2}{2}\right) \tag{17}$$

The crystal potential is obtained by summing up the potential of individual atoms with regard of their location in crystal lattice

$$U(\mathbf{r}) = \sum_i \varphi(\mathbf{r}-\mathbf{r}_i) = \frac{1}{(2\pi)^3}\int g(\mathbf{p})\sum_i \exp[i(\mathbf{p}(\mathbf{r}-\mathbf{r}_i))]d^3p$$
$$= \frac{1}{l^3}\int g(\mathbf{p})S(\mathbf{p})e^{-i\mathbf{pr}}\sum_{\mathbf{q}}\delta(\mathbf{p}-\mathbf{q})d^3p = \frac{1}{l^3}\sum_{\mathbf{q}} S(\mathbf{q})g(\mathbf{q})e^{-i\mathbf{qr}}$$
$$= \frac{1}{l^3}\sum_{\mathbf{q}} S(\mathbf{q})g_a(\mathbf{q})\exp\left(-\frac{u_1^2\mathbf{q}^2}{2} - i\mathbf{qr}\right) \equiv \sum_{\mathbf{q}} G(\mathbf{q})e^{-i\mathbf{qr}}, \tag{18}$$

where $\mathbf{q} = 2\pi(n_1, n_2, n_3)/l,\ l$ is the lattice constant, $S(\mathbf{q})$ is the structure factor.

Crystal potential

The motion of relativistic particles traveling along the family of crystal planes which characterized by Miller indices (hkm) is mainly governed by the continuum potential of these planes which can be obtained from Eq.(18) by letting

$$\mathbf{q} = \frac{2\pi n(hkm)}{l} \tag{19}$$

Than the sum (over n) in Eq.(18) becomes one-dimensional, giving the corresponding one-dimensional periodic continuum potential. Similarly, while traveling along the axes $< hkm >$ the relativistic particles experience, mainly, the continuum potential of these axes. The potential of an isolated atomic chain (axis) which is dominant at small distances $r_\perp = \varrho \ll l$ to this chain can be obtained from Eq.(18) if we integrate over $\mathbf{q}_\perp$ instead of the corresponding summation

$$\sum_{\mathbf{q}} \to \frac{s}{(2\pi)^2} \int d^2 q_\perp ;$$

$$U(\varrho) = \frac{1}{(2\pi)^2 d} \int \exp\left(-\frac{u_1^2 \mathbf{q}_\perp^2}{2} - i\mathbf{q}_\perp \boldsymbol{\varrho} \right) g_a(\mathbf{q}_\perp) d^2 q_\perp , \quad \varrho \ll l, \tag{20}$$

where s is the area per one chain, d is the mean distance between atoms in the chain. Substituting the particular representation of the function $g_a(\mathbf{q}_\perp)$ one can find the potential $U(\varrho)$.

In applications, the continuum axes potential can be conveniently represented in whole area per one axis by a simple expression which reproduces the behavior of the potential close to axis. We are using the following form of the continuum axes potential

$$U(x) = V_0 \left[\ln\left(1 + \frac{1}{x+\eta}\right) - \ln\left(1 + \frac{1}{x_0+\eta}\right)\right], \tag{21}$$

where

$$V_0 \simeq \frac{Ze^2}{d}, \quad x_0 = \frac{1}{\pi d n_a a_s^2}, \quad \eta \simeq \frac{2u_1^2}{a_s^2}, \quad x = \frac{\varrho^2}{a_s^2}, \tag{22}$$

Here ϱ is the distance from axis, u_1 is the amplitude of thermal vibration, d is the mean distance between atoms forming the axis, a_s is the effective screening radius of the potential, n_a is the mean atom density. Actually, the parameters of the potential were determined by means of fitting procedure.

The Radiation Process in Crystal

Radiation at Quasiperiodical Motion

In the system traveling with the averaged particle velocity this motion is $periodical$ by definition. It's convenient to analyze the radiation process in co-moving system where the mean particle's velocity is zero and then carry inverse Lorenz transformation. In co-moving system the properties of radiation depends strongly on the relation between the kinetic energy and the rest macc of the particle. As is known, in the nonrelativistic limit the radiation is dipole and determined completely by the Fourier components of the velocity. In this case, as a rule, one or a few first harmonics, which are multiples of the frequency of the particle motion, are emitted. Transforming back to the laboratory frame we obtain for the frequency of radiated photon (The Doppler effect)

$$\omega \simeq \frac{n\omega_0}{1 - \mathbf{nV}} \simeq \frac{2\gamma^2 n\omega_0}{1 + \gamma^2\vartheta^2} \qquad (23)$$

where ω_0 is the frequency of particle's motion in the laboratory frame, ϑ is the photon emission angle with respect to mean velocity $\mathbf{V}$ of particle, n is the number of the harmonics, $\mathbf{n} = \mathbf{k}/\omega$, $\mathbf{k}$ is the photon wave vector.

When the motion in the co-moving frame becomes relativistic, the nature of radiation process changes. First, the higher harmonics turns out to be essential in the radiation, and second, it becomes necessary to take into account the dependence of the particle longitudinal velocity (in the direction of mean velocity) on its transverse motion. Indeed, with an accuracy to terms $\sim U/\varepsilon$ (U is a potential in which the particle moves) $\gamma = const.$ and by definition $\gamma^2(1 - v^2) = \gamma^2(1 - v_\parallel^2 - v_\perp^2) = 1$, so for the longitudinal $v_\parallel$ and for mean V

velocities of particle we have

$$v_{\parallel} \simeq 1 - \frac{1 + v_{\perp}^2 \gamma^2}{2\gamma^2}, \quad V \simeq 1 - \frac{1 + \bar{v}_{\perp}^2 \gamma^2}{2\gamma^2} \tag{24}$$

Here it is assumed that $|\mathbf{v}_{\perp}| \ll V$. Taking this into account the expression for the frequency of radiation becomes

$$\omega \simeq \frac{2\gamma^2 n \omega_0}{1 + \gamma^2 \vartheta^2 + \varrho/2}, \quad \varrho = 2\gamma^2 \bar{v}_{\perp}^2, \tag{25}$$

where $\bar{v}_{\perp}^2$ is the mean square of the transverse velocity of particle.

In the ultrarelativistic limit of the motion in the co-moving frame ($\gamma v_{\perp} \gg 1$) the main contribution to the radiation is given by high harmonics with $n \gg 1$. The radiation spectrum is quasicontinuous and known formulae which describe magnetic bremsstrahlung hold. In this limit the radiation if form on a short section of the trajectory during the time $\tau \sim 1/|\dot{\mathbf{v}}|\gamma$, and the characteristic frequency of radiation (with allowance for the Doppler effect) is $\omega \sim |\dot{\mathbf{v}}|\gamma^3$. Taking into account that the main contribution to the radiation is made by the angles $\vartheta \sim v_\perp$, the estimate for n is

$$n \sim \frac{v_\perp^2 |\dot{\mathbf{v}}|\gamma^3}{\omega_0} \simeq (v_\perp \gamma)^3 = \left(\frac{\varrho}{2}\right)^{3/2} \tag{26}$$

Thus, the parameter ϱ entered in Eq.(23) characterizes the multipolarity of radiation. At $\varrho \ll 1$ the radiation is a dipole character, at $\varrho \sim 1$ a noticeable contribution is given by high harmonics and at $\varrho \gg 1$ we have the magnetic bremsstrahlung limit.

Motion of charged particle in a crystal

Consider the motion of a charged particle in the potential $U(\mathbf{r})$. If the vector of the initial velocity of a relativistic particle is not too close the direction of the axis (or plane), its motion, in accordance with the estimate of the potential of the potential depth V_0 (see Eqs.(21),(22), can be described in terms of the perturbation theory in crystal potential. The first order of such approach is the rectilinear trajectory approximation. Remember that the acceleration of relativistic particle is almost transverse

$$\dot{\mathbf{v}} \simeq \dot{\mathbf{v}}_{\perp} = -\frac{1}{\varepsilon}\frac{\partial U(\mathbf{r})}{\partial \mathbf{r}_{\perp}} \tag{27}$$

By substitution into right-hand side of Eq.(27) the coordinate $\mathbf{r}$ in the form $\mathbf{r} = \mathbf{r}_0 + \mathbf{v}_0 t$, making use of Eq.(18) and integrating with respect to time t, we obtain

$$\mathbf{v}(t) = \mathbf{v}_0 - \frac{1}{\varepsilon}\sum_{\mathbf{q}} \frac{\mathbf{q}_{\perp}}{q_{\parallel}} G(\mathbf{q}) e^{-i\mathbf{q}\mathbf{r}_0 - iq_{\parallel}t}, \tag{28}$$

where $q_{\|} = \mathbf{q}\mathbf{v}_0,\ \mathbf{q}_\perp = \mathbf{q} - \mathbf{v}_0(\mathbf{q}\mathbf{v}_0), 1 - |\mathbf{v}_0| \ll 1.$ Equation (28) describes the transverse particle motion characterized by the set of frequencies $\omega_0(\mathbf{q}) \equiv q_{\|}(\mathbf{q})$ (see Eq.(18)). If the angle ϑ_0 between the particle velocity and some crystal axis $\mathbf{e}_z$ is small, we have for the vectors $\mathbf{q}$ satisfying the condition $\mathbf{q}\mathbf{e}_z = 0$

$$q_{\|} \simeq \boldsymbol{\vartheta}_0 \mathbf{q}_\perp \tag{29}$$

where $\boldsymbol{\vartheta}_0 = \mathbf{v}_0 - \mathbf{e}_z(\mathbf{v}_0\mathbf{e}_z)$, whilst the remaining vectors $\mathbf{q}$ we have $q_{\|} \sim q_{\perp}$ and their contribution to the sum in Eq.(28) is negligible. Then from Eq.(28) we obtain $(G(\mathbf{q}) \sim V_0)$ an estimate of the variation of the transverse particle velocity

$$|\Delta \mathbf{v}| \sim \frac{V_0}{\varepsilon} \left| \frac{\mathbf{q}_\perp}{\boldsymbol{\vartheta}_0 \mathbf{q}_\perp} \right| . \tag{30}$$

When $\boldsymbol{\vartheta}_0 \mathbf{q}_\perp \sim \vartheta_0 q_\perp$ (for the axial case) we have the estimate $|\Delta \mathbf{v}| \sim V_0/(\varepsilon \vartheta_0)$. The rectilinear trajectory approximation as well as Eq.(28) are valid if the condition

$|\Delta\mathbf{v}| \ll \vartheta_0$ is fulfilled, and we obtain

$$\frac{|\Delta\mathbf{v}|}{\vartheta_0} \sim \frac{V_0}{\varepsilon\vartheta_0^2} \ll 1 \tag{31}$$

If the opposite inequality holds, the special type of motion, called channeling takes place. It is determined by the continuum axes potential.

Qualitative Consideration of Radiation in Crystal

The motion in crystal and, consequently, the parameter ϱ depends on the angle ϑ_0 (the angle with respect to chosen axis at which a particle is incident on a crystal) compared with the characteristic channeling angle (the Lindhard angle) $\vartheta_c \equiv (2V_0/\varepsilon)^{1/2}$, where V_0 is the scale of continuous potential of an axis (or a plane). If angles of incidence are in range $\vartheta_0 \leq \vartheta_c$, the electrons falling on crystal are captured into channels or low above-barrier states, whereas for $\vartheta_0 \gg \vartheta_c$ the incident particles move high above barrier. In latter case one can describe the motion using the approximation of a rectilinear trajectory. In this case as we just estimated $\bar{v}_\perp \sim V_0/\varepsilon\vartheta_0$, so that

$$\varrho(\vartheta_0) \simeq \left(\frac{2V_0}{m\vartheta_0}\right)^2 \qquad (32)$$

For angles of incidence in the range $\vartheta_0 \leq \vartheta_c$ the transverse (relative to axis) velocity of particles is $v_\perp \leq \vartheta_c$ and the parameter ϱ obeys $\varrho \leq \varrho_c$, where

$$\varrho_c = \frac{2V_0\varepsilon}{m^2}. \qquad (33)$$

It is clear from Eq.(32) that the problem has another characteristic angle $\vartheta_v = V_0/m$ so that $\varrho_c = (2\vartheta_v/\vartheta_c)^2$.

Description of Photon Emission in a Single Crystal at High Energy

We will the use the developed formalism which is valid for all types of external fields, including inhomogeneous and alternating fields, for description of radiation process in single crystal. So the probability of radiation can be written in the form

$$dw = \frac{\alpha}{(2\pi)^2}\frac{d^3k}{\omega}\int dt_1 \int dt_2 R_2^* R_1 \exp\left[\frac{\varepsilon}{\varepsilon - \hbar\omega}(kx_2 - kx_1)\right], \quad (34)$$

For unpolarized electrons and photons

$$R_2^* R_1 \to \frac{\varepsilon^2 + \varepsilon'^2}{2\varepsilon'^2}(\mathbf{v}_1\mathbf{v}_2 - 1) + \frac{(\hbar\omega)^2}{2\gamma^2\varepsilon'^2}. \quad (35)$$

The spectral distribution of the radiation intensity obtained from this formula has the form

$$dI(u) = \frac{e^2 m^2}{\pi\sqrt{3}\hbar^2}\frac{udu}{(1+u)^3}\left\{\left[\frac{1}{1+u} + (1+u)\right] K_{2/3}\left(\frac{2u}{3\chi}\right) - \int\limits_{2u/3\chi}^{\infty} K_{1/3}(y)dy\right\} \quad (36)$$

Below we assume that a crystal is thin and the condition $\varrho \gg 1$ is fulfilled. This means that in the range where the trajectories are essentially rectilinear ($\vartheta_0 \geq \vartheta_c, v_\perp \geq \vartheta_c$) the radiation mechanism is of the magnetic bremsstrahlung nature and the characteristics of radiation can be expressed in terms of local parameters of motion (as in Eqs.(34),(35)). The averaging procedure can be carried out simply if we know the distribution function in the transverse phase space $dN(\boldsymbol{\varrho}, \mathbf{v}_\perp)$ which for a thin crystal is defined directly by a initial conditions of incidence of particles on a crystal. For a given of incidence ϑ_0 we have $dN/N = d^3rF(\mathbf{r}, \vartheta_0)/V$, where N is the total number of particle V is the crystal volume. In the axial case the function $F(\mathbf{r}, \vartheta_0)$ is

$$F_{ax}(\boldsymbol{\varrho}, \vartheta_0) = \int \frac{d^2\varrho_0}{S(\varepsilon_\perp(\boldsymbol{\varrho}_0))}\vartheta(\varepsilon_\perp(\boldsymbol{\varrho}_0) - U(\boldsymbol{\varrho}), \quad (37)$$

where $U(\boldsymbol{\varrho})$ is the continuous potential of axis dependent on the transverse coordinate $\boldsymbol{\varrho}$ normalized so that $U(\boldsymbol{\varrho}) = 0$ on the boundary of cell;

$$S(\varepsilon_\perp(\boldsymbol{\varrho}_0)) = \int \vartheta(\varepsilon_\perp(\boldsymbol{\varrho}_0) - U(\boldsymbol{\varrho})d^2\varrho, \quad \varepsilon_\perp(\boldsymbol{\varrho}_0) = \frac{\varepsilon\vartheta_0^2}{2} + U(\boldsymbol{\varrho}_0), \tag{38}$$

$\vartheta(x)$ is the Heaviside function.

The further simplifications are possible when the potential of axis can be regarded as axially symmetric. In any case this is true at distances from the axis $\leq a_s$ where the electric field is maximal. If $U = U(\boldsymbol{\varrho}^2) = U(y)$ then

$$y_0(\varepsilon_\perp(y)) = \int\limits_0^{x_0} \vartheta(\varepsilon_\perp(y)) - U(x))dx, \quad \varepsilon_\perp(y)) = \frac{\varepsilon\vartheta_0^2}{2} + U(y) \tag{39}$$

Here we consider radiation when the angle of incidence $\vartheta_0 \ll V_0/m$. In the potential

Eq.(21) the electric field is strongly depends on the distance from axis (x):

$$\chi(x) = \frac{w(x)}{m}\gamma^2, \quad w = -\frac{1}{\varepsilon}\frac{\partial U}{\partial \varrho}, \quad \frac{\partial U}{\partial \varrho} = \frac{dU}{dx}\frac{\partial x}{\partial \varrho} = \frac{dU}{dx}\frac{2\sqrt{x}}{a_s} \tag{40}$$

As a result one obtains

$$\lambda = \lambda(x) = \frac{2u}{3\chi(x)} = \frac{u}{3\sqrt{x}g(x)\chi_s}, \quad g(x) = \frac{1}{(1+x+\eta)(x+\eta)}, \quad \chi_s = \frac{V_0\varepsilon}{a_s m^3} \tag{41}$$

Taking into consideration the previous analysis we obtain expression for radiation intensity in oriented crystal

$$dI^F(\omega) = \frac{\alpha m^2}{\pi\sqrt{3}\hbar^2}\frac{\omega d\omega}{\varepsilon^2}\int\limits_0^{x_0}\frac{dx}{x_0}\Bigg\{\int\limits_0^{x_0}\frac{dy}{y_0(\varepsilon_\perp(y))}\vartheta(\varepsilon_\perp(y)) - U((x))$$

$$\times\left[\varphi(\varepsilon)K_{2/3}(\lambda) - \int\limits_\lambda^\infty K_{1/3}(y)dy\right]\Bigg\}, \quad \varphi(\varepsilon) = \frac{\varepsilon}{\varepsilon'} + \frac{\varepsilon'}{\varepsilon}. \qquad (42)$$

Originally the radiation process was considered basing on analysis of expressions with includes decomposition of the crystal potential of the type Eqs.(18)-(21). The correction to the intensity was found. It is proportional to $(m\vartheta_0/V_0)^2$. The energy dependence of $I^F(\varepsilon)/\varepsilon = L_{ch}^{-1}$ is shown in Fig.1,3.

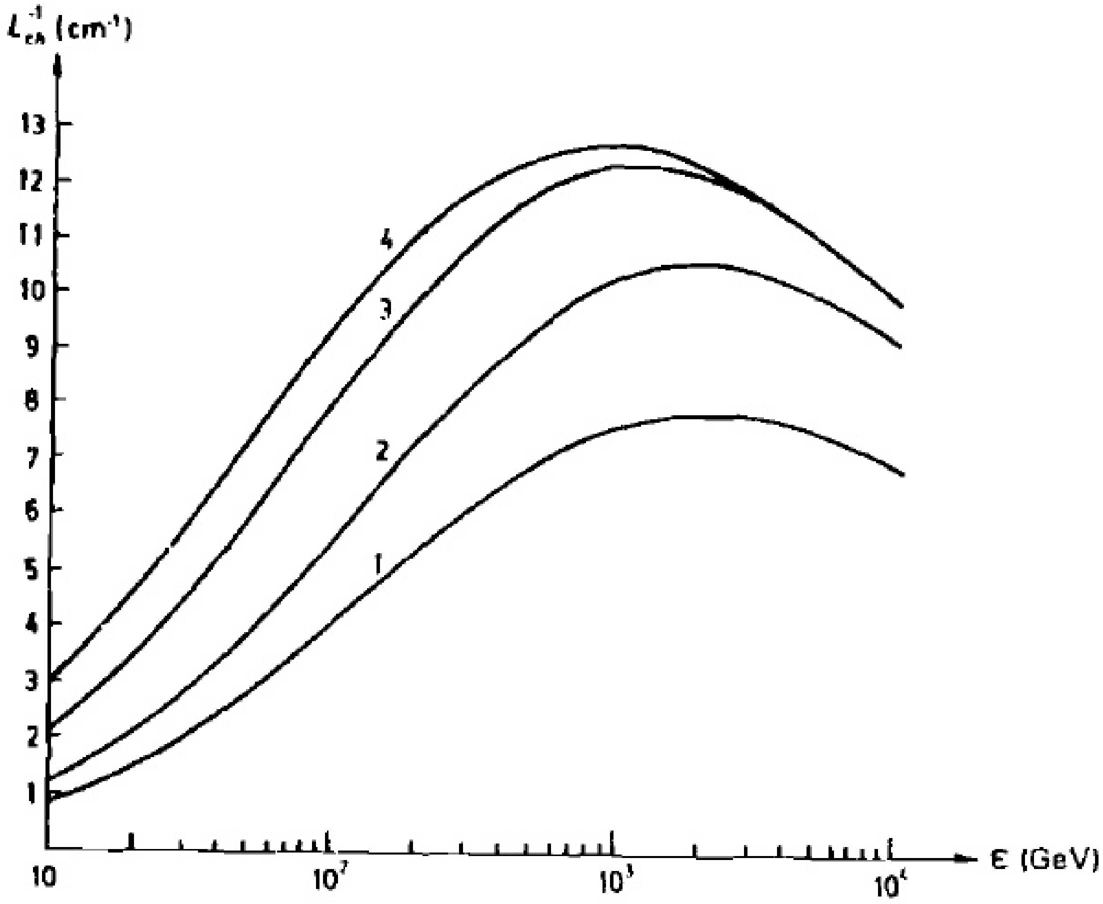

Figure 1: The energy dependence of inverse crystal radiation length for Si, axis $< 110 >$ at T=293 K (curve 1), for diamond axis $< 111 >$ at T=293 K (curve 2), for Ge axis $< 110 >$at T=280 K (curve 3), for Ge axis $< 110 >$ at T=100 K (curve 4). For amorphous materials L_{am}^{-1}=0.059 (C), 0.11 (Si), 0.43 (Ge).

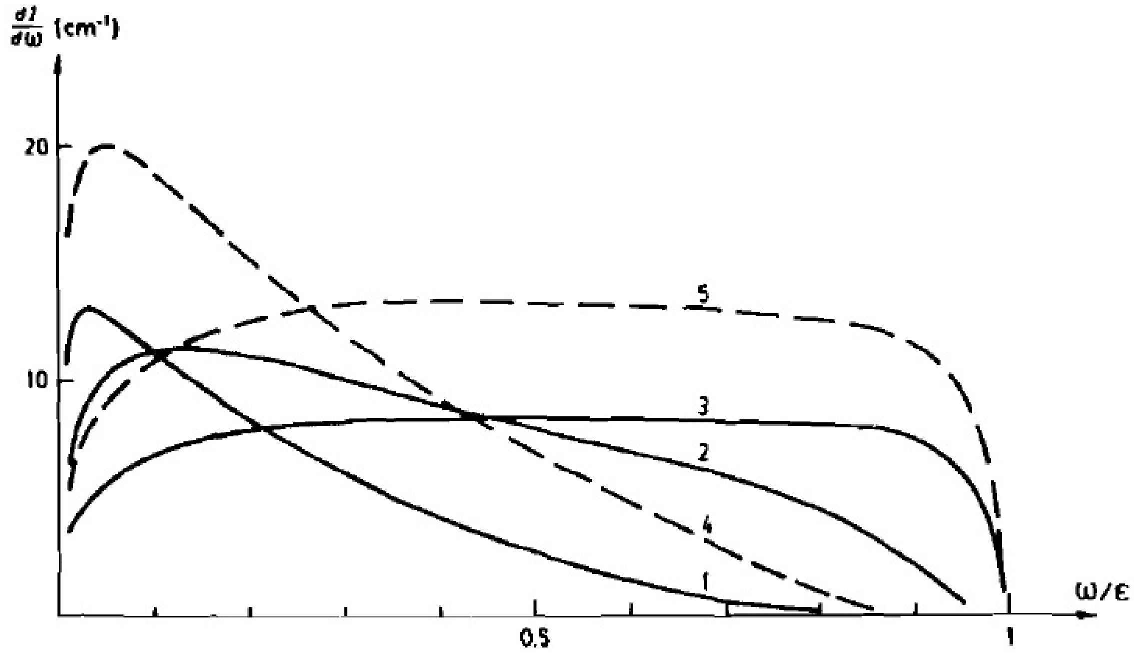

Figure 2: The spectral intensity of radiation at a given energy for Si $< 110 >$ at T=293 K (solid curves) for $\varepsilon = 100$ GeV (curve 1), for $\varepsilon = 700$ GeV (curve 2), for $\varepsilon = 5$ TeV (curve 3), and for Ge $< 110 >$ at T=280 K (dashed curves):$\varepsilon = 100$ GeV (curve 4), for $\varepsilon = 5$ TeV (curve 5)

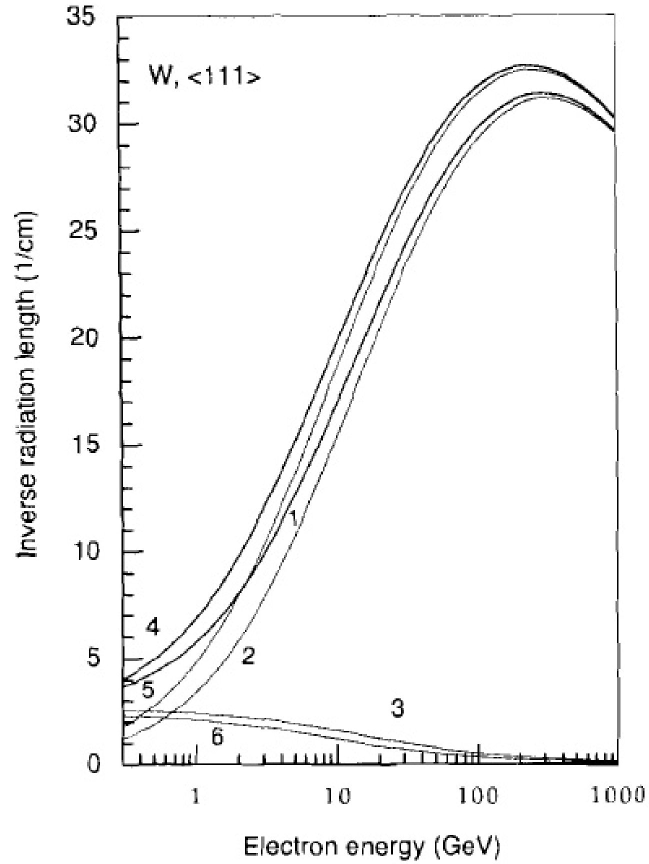

Figure 3: The inverse radiation length in tungsten, axis $< 111 >$ at different temperatures T vs electron initial energy. Curves 1 and 4 present the total effect $I^F(\varepsilon)/\varepsilon = L_{ch}^{-1}$ for T=293 K and T=100 K; correspondingly curves 2 and 5 give the coherent contribution and curves 3 and 6 give the incoherent contribution,L_{am}^{-1}=2.9.

Description of Pair Creation by a Photon in a Single Crystal at High Energy

The probability of pair creation can be found from Eq.(42)(written for probability:$dI = \hbar\omega dW$) omitting factor connected with particle distribution in the transverse plane, taking into account that the final state is changed:$d^3k \to d^3p(\omega^2 d\omega \to \varepsilon^2 d\varepsilon)$ and making substitution: $\omega \to -\omega; \varepsilon \to -\varepsilon$. Than the combination entering in λ will be

$$\frac{\omega}{\varepsilon(\varepsilon-\omega)} \to \frac{-\omega}{-\varepsilon(-\varepsilon+\omega)} = \frac{\omega}{\varepsilon(\omega-\varepsilon)} \tag{43}$$

As a result one obtains

$$dW_p(\omega) = \frac{\alpha m^2}{\pi\sqrt{3}\hbar^2\omega^2}\int\limits_0^{x_0}\frac{dx}{x_0}\left\{\int\limits_0^{\omega} d\varepsilon\left[\varphi(\varepsilon)K_{2/3}(\lambda) - \int\limits_{\lambda}^{\infty}K_{1/3}(y)dy\right]\right\}. \tag{44}$$

The spectrum of particles of the created pair follows from Eq.(44) if one omits the integration over ε. For the axis $< 111 >$ in tungsten for T=293 K for various photon energies it is shown in Fig.5.

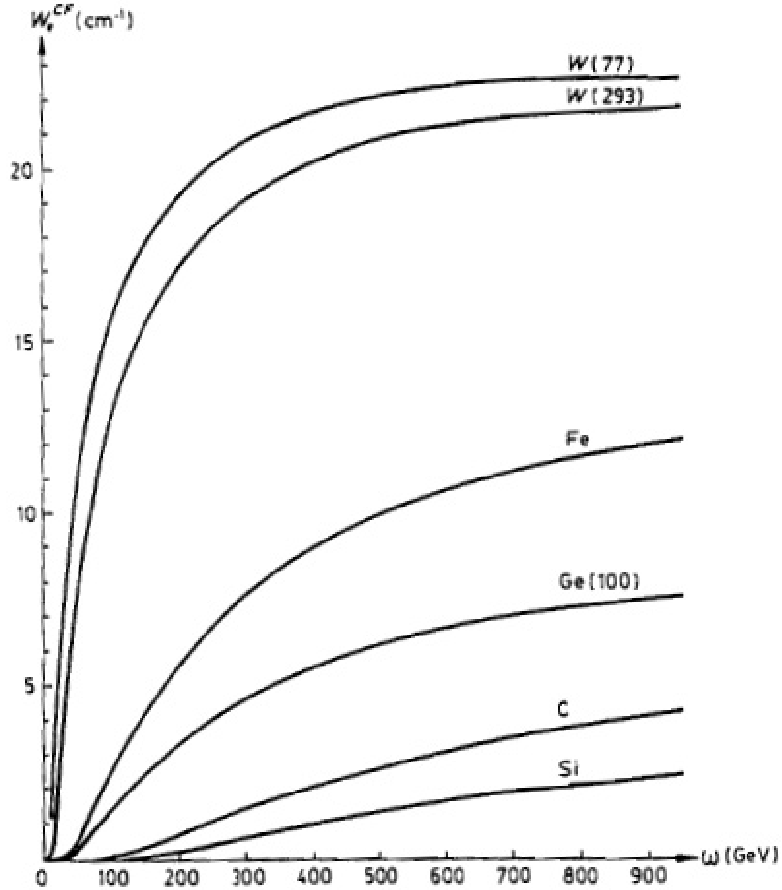

Figure 4: Probability of pair production by a photon for the angle of incidence $\vartheta_0 = 0$ with respect $< 111 >$ axis (for Ge $< 110 >$ axis). The numbers in parentheses denote temperature, where it not given, T=293 K.

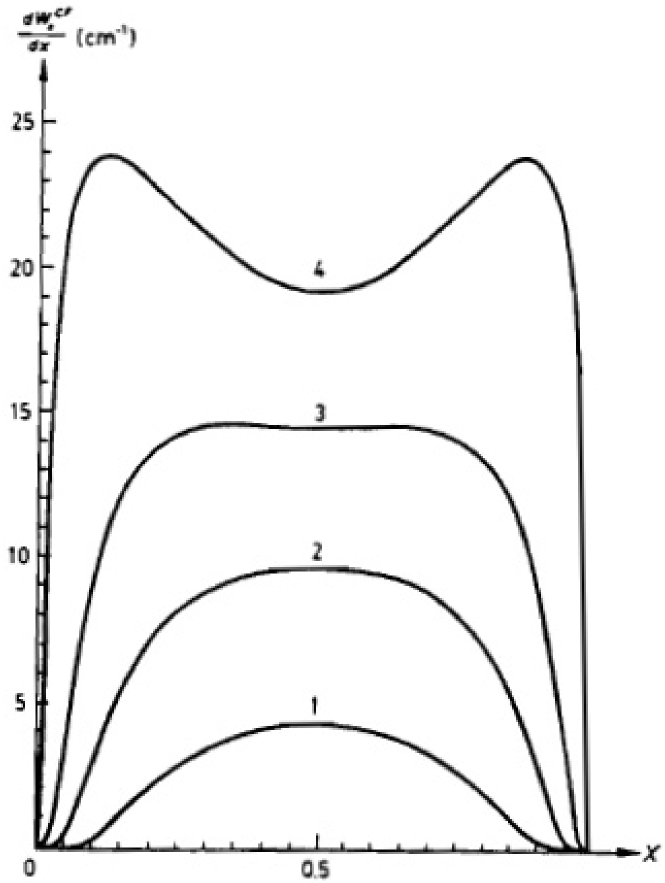

Figure 5: Probability of pair production by a photon for the angle of incidence $\vartheta_0 = 0$ with respect $< 111 >$ axis in W, T=293 K, $\vartheta_0 = 0$, ω (in GeV): $\omega = 25$ (curve 1), $\omega = 50$ (curve 2), $\omega = 100$ (curve 3), $\omega = 500$ (curve 4)

Comparison of Theory and Data

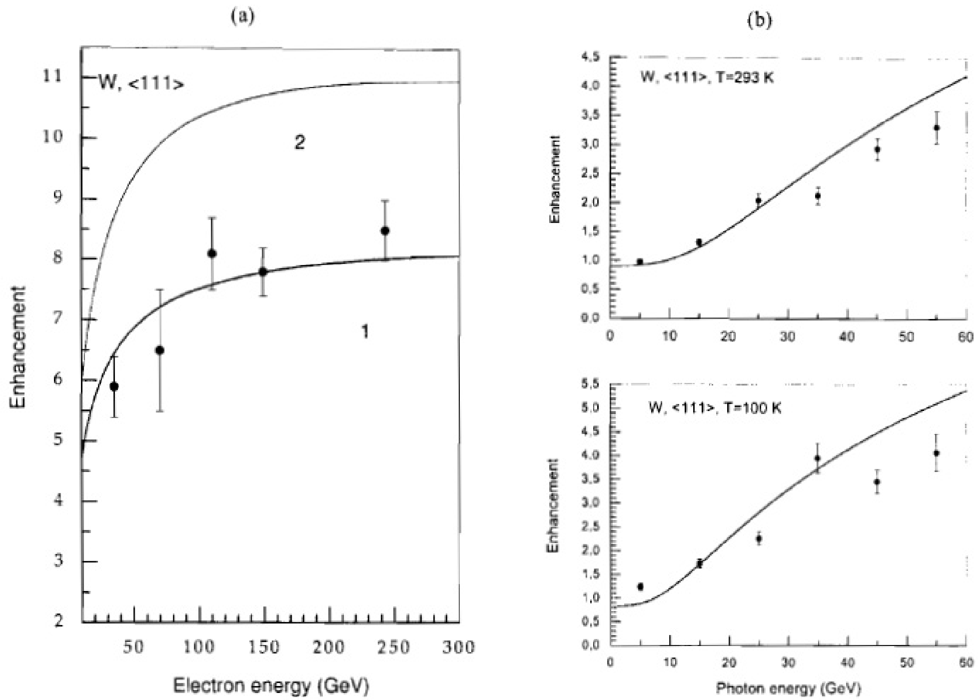

Figure 6: Comparison of theory and experiment.(a) Enhancement of radiation intensity in tungsten $< 111 >$ axis, T=293 K. The curve is for target with thickness $l = 200\mu m$, where energy loss is taken into account. The curve 2 is for considerably thinner target, where one can neglect energy loss. Data from "K.Kirsenom, U.Mikkelsen, E.Uggerhoj $at\ all$, PRL, 2001." (b) Enhancement of the pair creation probability for different temperatures, $< 111 >$ axis. Data from "K.Kirsenom, Yu.V.Kononets, U.Mikkelsen, $at\ all$, NIM B, 1998."

Coherent and Incoherent Pair Creation and Radiation in Oriented Crystal

For axial orientation of a crystal the ratio of the atom density $n(\varrho)$ in the vicinity of the axis to the mean atom density n_a is

$$\frac{n(\varrho)}{n_a} = \frac{e^{-\varrho^2/2u_1^2}}{2\pi d n_a u_1^2}, \quad \frac{n(0)}{n_a} \sim \left(\frac{d}{u_1}\right)^2, \quad \omega_0 = \omega_e \frac{n_a}{n(0)}, \tag{45}$$

where ϱ is the distance from the axis, u_1 is the amplitude of the thermal vibrations, d is the mead distance between atoms forming the axis. The strength of the electric field in the vicinity of the axis is

$$F \sim \frac{2V_0\varrho}{\varrho^2 + 2u_1^2}, \quad F_{max} \sim \frac{V_0}{u_1}, \quad V_0 \simeq \frac{Ze^2}{d}. \tag{46}$$

The formation length of pair creation in the field is $l_m \sim m/F_{max}$. The ratio of l_f and l_m is

$$\frac{l_f}{l_m} \sim \frac{V_0}{mu_1}\frac{\omega}{m^2} \sim \frac{\omega}{\omega_m} \equiv \kappa_m \sim 1, \quad l_f \sim l_m. \tag{47}$$

It is useful to compare the characteristic energy ω_0 with energy ω_m for which the probability of pair creation in the field becomes comparable with the Bethe-Maximon probability.

Here we consider the case when angle of incidence $\vartheta_0 \ll V_0/m$. This is the condition that the distance $\varrho \sim u_1$ as well as the atom density and the transverse field of axis can be considered as constant over the formation length

$$\frac{\Delta\varrho}{u_1} = \frac{\vartheta_0 l_f}{u_1} \sim \frac{\vartheta_0 l_m}{u_1} \ll 1. \qquad (48)$$

The general expression for the spectral distribution of particles of pair created by a photon

$$dW(\omega, y) = \frac{\alpha m^2}{2\pi\omega}\frac{dy}{y(1-y)}\int_0^{x_0}\frac{dx}{x_0}G(x,y), \quad G(x,y) = \int_0^{\infty}F(x,y,t)dt + s_3\frac{\pi}{4},$$
$$F(x,y,t) = \mathrm{Im}\left\{e^{f_1(t)}\left[s_2\nu_0^2(1+ib)f_2(t) - s_3f_3(t)\right]\right\}, \quad b = \frac{4\kappa_1^2}{\nu_0^2}, \quad y = \frac{\varepsilon}{\omega},$$
$$f_1(t) = (i-1)t + b(1+i)(f_2(t)-t), \quad f_2(t) = \frac{\sqrt{2}}{\nu_0}\tanh\frac{\nu_0 t}{\sqrt{2}},$$
$$f_3(t) = \frac{\sqrt{2}\nu_0}{\sinh(\sqrt{2}\nu_0 t)}, \tag{49}$$

where

$$s_2 = y^2+(1-y)^2,\ s_3 = 2y(1-y),\ \nu_0^2 = 4y(1-y)\frac{\omega}{\omega_c(x)},\ \kappa_1 = y(1-y)\kappa(x), \tag{50}$$

ε is the energy of one of the created particles.

The situation is considered when the photon angle of incidence ϑ_0 (the angle between photon momentum **k** and the axis (or plane)) is small $\vartheta_0 \ll V_0/m$.

Parameters of the pair photoproduction and radiation processes in the tungsten crystal, axis $< 111 >$ and the germanium crystal, axis $< 110 >$ for two temperatures T
$(\varepsilon_0 = \omega_0/4, \varepsilon_m = \omega_m, \varepsilon_s = \omega_s)$

Crystal	T(K)	V_0(eV)	x_0	η_1	η	ω_0(GeV)	ε_m(GeV)	ε_s(GeV)	h
W	293	417	39.7	0.108	0.115	29.7	14.35	34.8	0.348
W	100	355	35.7	0.0401	0.0313	12.25	8.10	43.1	0.612
Ge	293	110	15.5	0.125	0.119	592	88.4	210	0.235
Ge	100	114.5	19.8	0.064	0.0633	236	50.5	179	0.459

For an axial orientation of crystal the ratio of the atom density $n(\varrho)$ in the vicinity of an axis to the mean atom density n_a is

$$\frac{n(x)}{n_a} = \xi(x) = \frac{x_0}{\eta_1} e^{-x/\eta_1}, \quad \omega_0 = \frac{\omega_e}{\xi(0)}, \quad \omega_e = 4\varepsilon_e = \frac{m}{4\pi Z^2 \alpha^2 \lambda_c^3 n_a L_0}. \qquad (51)$$

The functions and values in Eqs. above are

$$
\begin{aligned}
&\omega_c(x) = \frac{\omega_e(n_a)}{\xi(x) g_p(x)} = \frac{\omega_0}{g_p(x)} e^{x/\eta_1}, \quad L = L_0 g_p(x), \\
&g_p(x) = g_{p0} + \frac{1}{6L_0}\left[\ln\left(1+\kappa_1^2\right) + \frac{6D_p\kappa_1^2}{12+\kappa_1^2}\right], \quad g_{p0} = 1 - \frac{1}{L_0}\left[\frac{1}{42} + h\left(\frac{u_1^2}{a^2}\right)\right], \\
&h(z) = -\frac{1}{2}\left[1 + (1+z)e^z \mathrm{Ei}(-z)\right], \quad L_0 = \ln(ma) + \frac{1}{2} - f(Z\alpha), \\
&a = \frac{111 Z^{-1/3}}{m}, \quad f(\xi) = \sum_{n=1}^{\infty} \frac{\xi^2}{n(n^2+\xi^2)},
\end{aligned}
\tag{52}
$$

where the function $g_p(x)$ determines the effective logarithm using the interpolation procedure, $D_p = D_{sc} - 10/21 = 1.8246$, $D_{sc} = 2.3008$ is the constant entering in the radiation spectrum at $\chi/u \gg 1$ (or in electron spectrum in pair creation process at $\kappa_1 \gg 1$), Ei(z) is the integral exponential function.

The expression for $dW(\omega, y)$ includes both the coherent and incoherent contributions as well as the influence of the multiple scattering (the LPM effect) on the pair creation process. The probability of the coherent pair creation is the first term ($\nu_0^2 = 0$) of the decomposition of Eq.(49) over ν_0^2

$$dW^{coh}(\omega, y) = \frac{\alpha m^2}{2\sqrt{3}\pi\omega}\frac{dy}{y(1-y)}\int_0^{x_0}\frac{dx}{x_0}\left[2s_2K_{2/3}(\lambda) + s_3\int_\lambda^\infty K_{1/3}(z)dz\right],$$
$$\lambda = \lambda(x) = \frac{2}{3\kappa_1}, \tag{53}$$

where $K_\nu(\lambda)$ is MacDonald's function. The probability of the incoherent pair creation is the second term ($\propto \nu_0^2$) of the mentioned decomposition

$$dW^{inc}(\omega, y) = \frac{4Z^2\alpha^3 n_a L_0}{15m^2}dy\int_0^\infty \frac{dx}{\eta_1}e^{-x/\eta_1}f(x, y)g_p(x), \tag{54}$$

where $g_p(x)$ is defined above

$$f(x,y) = f_1(z) + s_2 f_2(z), \quad f_1(z) = z^4\Upsilon(z) - 3z^2\Upsilon'(z) - z^3,$$
$$f_2(z) = (z^4 + 3z)\Upsilon(z) - 5z^2\Upsilon'(z) - z^3, \quad z = z(x,y) = \kappa_1^{-2/3}. \tag{55}$$

Here

$$\Upsilon(z) = \int_0^\infty \sin\left(zt + \frac{t^3}{3}\right) dt \tag{56}$$

is the Hardy function.

For $\omega = 7$ GeV the interplay of the coherent and incoherent contributions is leading to the nearly flat final spectrum (the variation is less than 10 %, this is quite unusual). It should be noted that for $\omega = 7$ GeV the right end ($y = 0.5$) is slightly lower than the left end of spectrum ($y \to 0$): $dW/dy(y \to 0) = 2.303\ \mathrm{cm}^{-1}$ and $dW(y = 0.5) = 2.215\ \mathrm{cm}^{-1}$, while the sum of the incoherent and coherent contributions is slightly higher: $dW^{inc}/dy(y = 0.5) + dW^{coh}/dy(y = 0.5) = 2.365\ \mathrm{cm}^{-1}$. The arising difference is the consequence of the LPM effect. This property may be very useful in experimental study.

The next terms of decomposition of the pair creation probability $dW = dW(\omega, y)$ over ν_0^2 describe the influence of multiple scattering on the pair creation process, the LPM effect. The third term ($\propto \nu_0^4$) of the mentioned decomposition has the form

$$\frac{dW^{(3)}(\omega, y)}{dy} = -\frac{\alpha m^2 \omega \sqrt{3}}{5600\pi \omega_0^2 x_0} \int_0^{x_0} \frac{g_p^2(x)}{\kappa(x)} \Phi(\lambda) e^{-2x/\eta_1} dx$$
$$\Phi(\lambda) = \lambda^2 \left(s_2 F_2(\lambda) - s_3 F_3(\lambda) \right),$$
$$F_2(\lambda) = (7820 + 126\lambda^2)\lambda K_{2/3}(\lambda) - (280 + 2430\lambda^2) K_{1/3}(\lambda),$$
$$F_3(\lambda) = (264 - 63\lambda^2)\lambda K_{2/3}(\lambda) - (24 + 3\lambda^2) K_{1/3}(\lambda), \tag{57}$$

where λ is defined above.

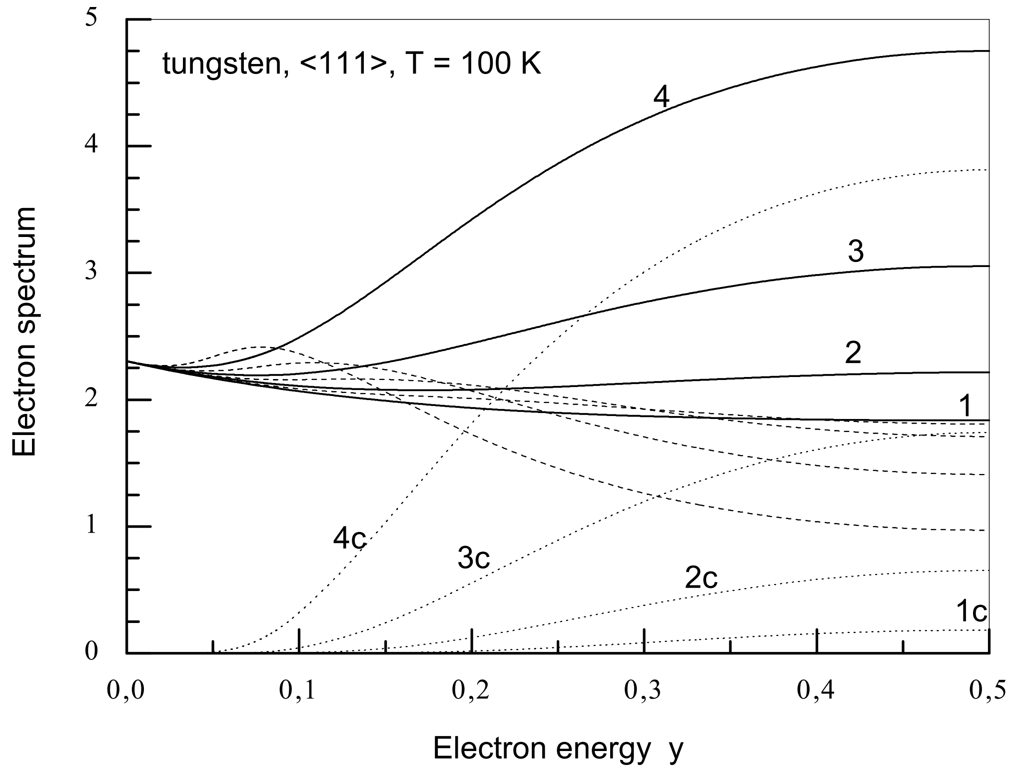

Figure 7: The spectral distribution of created by a photon pair (in units cm^{-1}) vs the electron energy $y = \varepsilon/\omega$ in tungsten, axis $< 111 >$, temperature T=100 K. (a) The curves 1, 2, 3, 4 are the theory prediction $dW(\omega, y)/dy$ for photon energies $\omega = 5,\ 7,\ 10,\ 15$ GeV respectively, The doted curves are the corresponding coherent contributions, the dashed curves present the incoherent contributions.

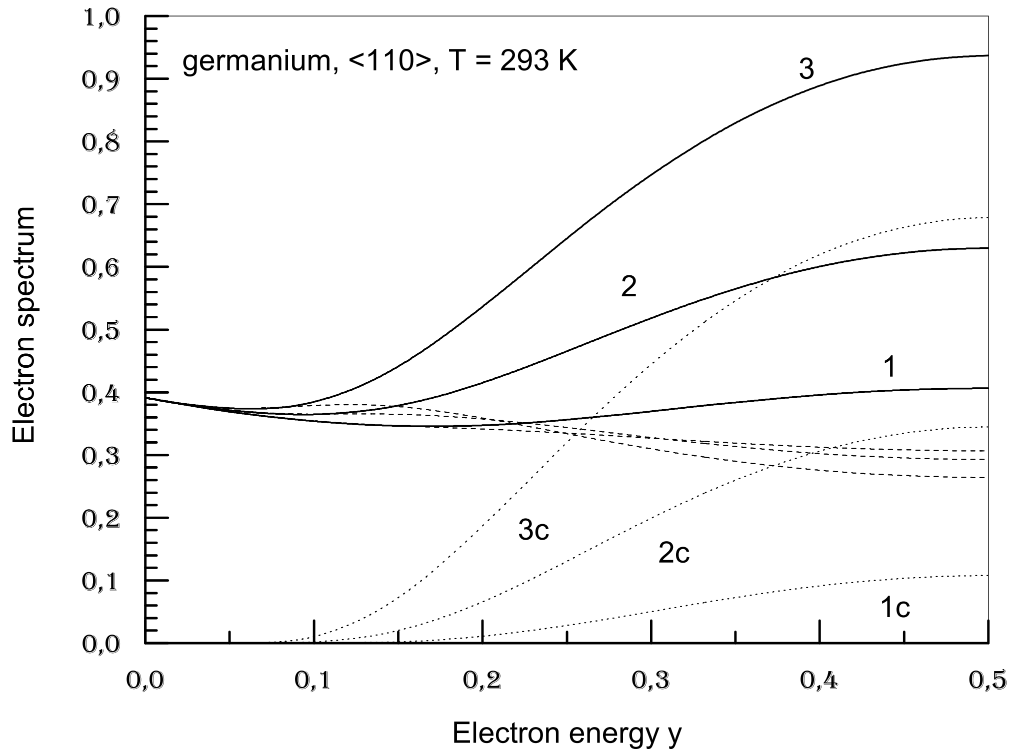

Figure 8: The spectral distribution of created by a photon pair (in units cm^{-1}) vs the electron energy $y = \varepsilon/\omega$ in germanium, axis $< 110 >$, temperature T=293 K. The curves 1, 2, 3, are the theory prediction $dW(\omega, y)/dy$ for photon energies $\omega = 55,\ 75,\ 95$ GeV respectively, The doted curves are the corresponding coherent contributions, the dashed curves present the incoherent contributions.

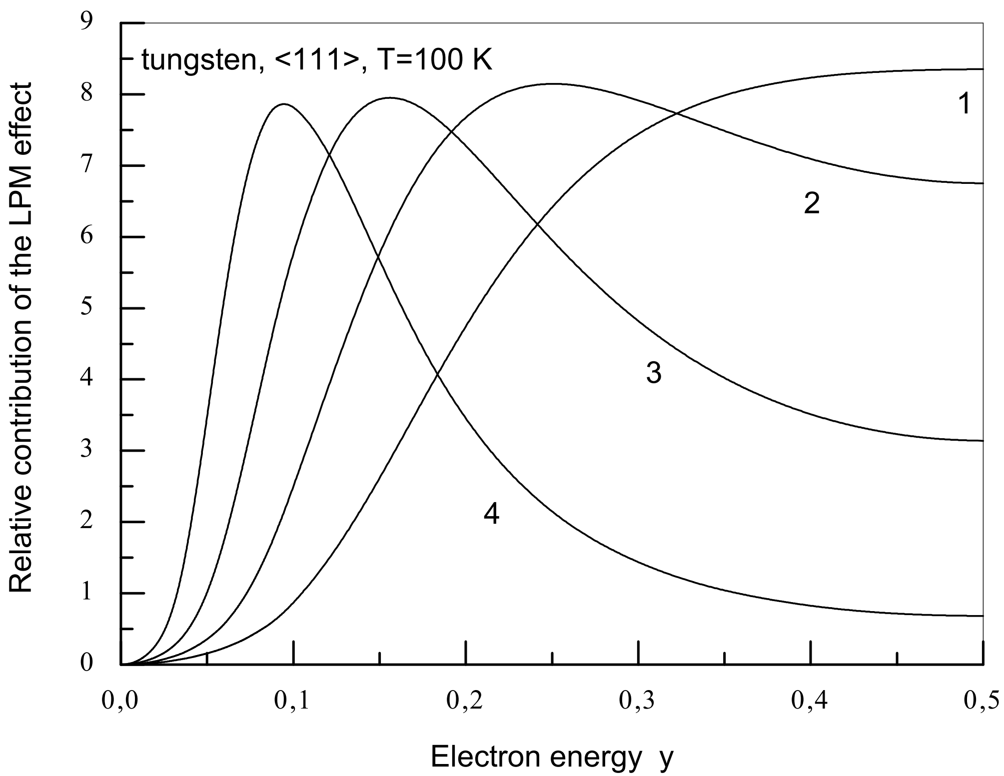

Figure 9: The relative contribution of the LPM effect in the spectral distribution of created electron (see Eq.(57)) $\Delta_p(\omega, y)$ (per cent). The curves 1, 2, 3, 4 are correspondingly for photon energies $\omega = 5,\ 7,\ 10,\ 15$ GeV, Δ_p^{max}=8.35 %.

Quantum Electrodynamics in External Field

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AARHUS UNIVERSITY NOVEMBER 11, 2009

Electromagnetic field in different frames

The electromagnetic field is given by the antisymmetric tensor of second rank

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -H_3 & H_2 \\ E_2 & H_3 & 0 & -H_1 \\ E_3 & -H_2 & H_1 & o \end{pmatrix} \qquad (1)$$

Here $\mathbf{E}(E_1, E_2, E_3)$ is the vector of the electric field and $\mathbf{H}(H_1, H_2, H_3)$ is pseudo-vector (axial vector) of the magnetic field. The Lorenz transformation of these fields can be written in the vector form

$$\mathbf{H} = \gamma\left[\mathbf{H}' - \frac{\mathbf{v}(\mathbf{v}\mathbf{H}')}{1+1/\gamma} + (\mathbf{v}\times\mathbf{E}')\right], \quad \gamma = \frac{1}{\sqrt{1-v^2}},$$

$$\mathbf{E} = \gamma\left[\mathbf{E}' - \frac{\mathbf{v}(\mathbf{v}\mathbf{E}')}{1+1/\gamma} - (\mathbf{v}\times\mathbf{H}')\right], \qquad (2)$$

where $\mathbf{v}$ is the velocity of prime system with respect to the initial one.

Let us find system where $\mathbf{H}'$ and $\mathbf{E}'$ are directed along one axis (we choose z-axis), the velocity $\mathbf{v}$, which is transverse to both vectors $\mathbf{H}$ and $\mathbf{E}$ is directed along x-axis. Then according with our condition $E'_x = H'_x = E'_y = H'_y = 0$ and

$$E_y = vH'_z\gamma, \quad E_z = E'_z\gamma, \quad H_y = -vE'_z\gamma, \quad H_z = H'_z\gamma,$$
$$E_x = E'_x = H_x = H'_x = 0 \tag{3}$$

One can see, that

$$(\mathbf{E} \times \mathbf{H})_x = E_yH_z - E_zH_y = v\gamma^2({E'_z}^2 + {H'_z}^2) = v\gamma^2(\mathbf{E'}^2 + \mathbf{H'}^2) = v(E_z^2 + H_z^2), \tag{4}$$

while two other components of the vector product $(\mathbf{E} \times \mathbf{H})$ vanish since they contained $E_x = H_x = 0$. this means that the vector product $(\mathbf{E} \times \mathbf{H})$ is directed along $\mathbf{v}$ so that

$$(\mathbf{E} \times \mathbf{H}) = \mathbf{v}\gamma^2(\mathbf{E'}^2 + \mathbf{H'}^2), \tag{5}$$

From the other side

$$\mathbf{E}^2+\mathbf{H}^2=E_y^2+E_z^2+H_y^2+H_z^2=\gamma^2(1+v^2)(\mathbf{E}'^2+\mathbf{H}'^2), \tag{6}$$

and we obtain finally

$$\frac{(\mathbf{E}\times\mathbf{H})}{\mathbf{E}^2+\mathbf{H}^2}=\frac{\mathbf{v}}{1+v^2}. \tag{7}$$

So we find velocity which from frame with arbitrary directed vectors $\mathbf{E}$ and $\mathbf{H}$ bring us to the special frame where these vectors are parallel to each other.

It is known that exist two combinations constructed from fields which are invariant with respect the Lorenz transformations:

$$F=\frac{1}{2}(\mathbf{E}^2-\mathbf{H}^2), \quad G=\mathbf{E}\mathbf{H} \tag{8}$$

In the special frame

$$H=\frac{G}{E}, \quad E^2-\frac{G^2}{E^2}=2F. \tag{9}$$

Solving the last equation one finds

$$E = \sqrt{\sqrt{F^2 + G^2} + F}, \quad H = \sqrt{\sqrt{F^2 + G^2} - F} \tag{10}$$

So it is often convenient to solve a problem in the special frame and result will presented in terms of invariants.

Classical Equation of Motion

As it is well known, the covariant equation of motion in external electromagnetic field $F^{\mu\nu}$ is

$$m\frac{du^\mu}{d\tau} = eF^{\mu\nu}u_\nu, \qquad (11)$$

where $u^\mu = dx^\mu/d\tau$ is the 4-velocity, τ is the proper time $(d\tau = dt/\gamma,\ \gamma = 1/\sqrt{1-v^2})$
Thus

$$u^1 = \frac{dx^1}{d\tau} = v_x\gamma \qquad (12)$$

and the components of 4-velocity are

$$u^\mu = (\gamma, \mathbf{v}\gamma). \qquad (13)$$

It is evident that

$$u^\mu u_\mu = (u^0)^2 - \mathbf{u}^2 = 1, \quad u^0 = \sqrt{1+\mathbf{u}^2} \qquad (14)$$

The equation Eq.(11) can be written in tensor form

$$m\frac{du}{d\tau} = eFu, \tag{15}$$

where F is the antisymmetric tensor of second rank, u is the 4-vector written as a column

$$u = \begin{pmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{pmatrix} \tag{16}$$

The formal solution of Eq.(15) for constant F is

$$u = \exp\left(\frac{e\tau F}{m}\right) u(0), \tag{17}$$

and in general case

$$u = \exp\left(\frac{e\int\limits_0^\tau F(\tau')d\tau'}{m}\right)u(0), \tag{18}$$

To analyze the result Eq.(17) we have to calculate $\exp(sF)$. For transformation of matrix expressions of this type is is convenient to use the special system introduced above. In this system the electromagnetic field tensor has the form

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & E \\ 0 & 0 & -H & 0 \\ 0 & H & 0 & 0 \\ -E & 0 & 0 & 0 \end{pmatrix} \tag{19}$$

And can be presented as

$$F_{\mu\nu} = C_{\mu\nu}E + B_{\mu\nu}H, \tag{20}$$

where

$$C_{\mu\nu} = g^0_\mu g^3_\nu - g^3_\mu g^0_\nu, \quad B_{\mu\nu} = g^2_\mu g^1_\nu - g^1_\mu g^2_\nu. \tag{21}$$

Here $g_{\mu\nu} = g^{\mu\nu}$ is the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{22}$$

The explicit form of this tensors is

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad B_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

and

$$C^2_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{24}$$

so that

$$C^2_{\mu\nu} - B^2_{\mu\nu} = g_{\mu\nu} \tag{25}$$

The tensors $C_{\mu\nu},\ B_{\mu\nu}$ satisfy the following relations

$$\begin{aligned} &(CB)_{\mu\nu} = 0, \quad (C^{2k+1})_{\mu\nu} = C_{\mu\nu}, \quad (C^{2k})_{\mu\nu} = C^2_{\mu\nu}, \\ &B^{2k+1}_{\mu\nu} = (-1)^k B_{\mu\nu}, \quad B^{2k}_{\mu\nu} = (-1)^{k+1} B^2_{\mu\nu}, \quad k \geq 1 \end{aligned} \tag{26}$$

The formulas Eqs.(19)-(26) enable us to expand an arbitrary function of $F_{\mu\nu}$ in terms of tensors $C_{\mu\nu},\ B_{\mu\nu},\ C^2_{\mu\nu},\ B^2_{\mu\nu}$ which are linearly independent and complete set (in the special system).

We will illustrate this on the example $\exp(sF)$:

$$(\exp(sF))_{\mu\nu} = \left(\sum_{k=0}^{\infty} \frac{(sF)^k}{k!}\right)_{\mu\nu}$$

$$= \sum_{k=0}^{\infty} \frac{s^{2k}}{(2k)!}(CE + BH)^{2k}_{\mu\nu} + \sum_{k=0}^{\infty} \frac{s^{2k+1}}{(2k+1)!}(CE + BH)^{2k+1}_{\mu\nu}$$

$$= g_{\mu\nu} + (C^2)_{\mu\nu} \sum_{k=1}^{\infty} \frac{(sE)^{2k}}{(2k)!} - (B^2)_{\mu\nu} \sum_{k=1}^{\infty} (-1)^k \frac{(sH)^{2k}}{(2k)!}$$

$$+(C)_{\mu\nu} \sum_{k=0}^{\infty} \frac{(sE)^{2k+1}}{(2k+1)!} + (B)_{\mu\nu} \sum_{k=0}^{\infty} (-1)^k \frac{(sH)^{2k+1}}{(2k+1)!}$$

$$= (C^2)_{\mu\nu} \cosh(sE) - (B^2)_{\mu\nu} \cos(sH) + (C)_{\mu\nu} \sinh(sE) + (B)_{\mu\nu} \sin(sH) \tag{27}$$

So, we obtain the explicit representation of four velocity Eq.(17) $u(s) = \exp(sF)u(0),\ s =$

$e\tau/m$. The coordinate can be found solving the equation

$$\frac{dx}{ds} = \frac{m}{e}\exp(Fs)u(0) \tag{28}$$

the solution of which is

$$x(s) = \frac{m}{e}\frac{(\exp(sF) - 1)}{F}u(0) \tag{29}$$

To obtain explicit form of solution let us seek it in the form

$$\frac{(\exp(sF) - 1)}{F} = \frac{C^2\cosh(sE) - B^2\cos(sH) + C\sinh(sE) + B\sin(sH) - C^2 + B^2}{CE + BH}$$
$$= \lambda_1 C^2 + \lambda_3 C + \lambda_2 B^2 + \lambda_4 B \tag{30}$$

Multiplying both parts of Eq.(30) by the denominator $CE + BH$ and equating the coefficients in front of each tensors (after using relations Eq.(26)) in both parts of Eq.(30), we obtain

$$\lambda_1 = \frac{\sinh(sE)}{E}, \quad \lambda_2 = -\frac{\sin(sH)}{H}, \quad \lambda_3 = \frac{\cosh(sE) - 1}{E}, \quad \lambda_4 = \frac{1 - \cos(sH)}{H}, \tag{31}$$

which is the solution of the problem.

Solution in magnetic field

We will illustrate the found result analyzing two particular cases:

In special frame we are using the field is directed along axis 3(z). The 4-vector x we will present as column (see C). Using Eqs.(29 and (31)) the solution can be written in the form

$$x = \frac{m}{e}\left(-\frac{\sin(sH)}{H}B^2 + \frac{1-\cos(sH)}{H}B\right)u(0), \tag{32}$$

where x is the 4-vector written as a column. Here operators (matrix B and B^2 Eqs.(23 and (24))) are acting on the transverse to the magnetic field components of 4-velocity, the longitudinal component ($u_z(0)$) remains constant. One can write the quantities entering in Eq.(32) as

$$u^1(0) = u_x(0) = -v_0\gamma\cos\beta, \quad u^2(0) = u_y(0) = -v_0\gamma\sin\beta,$$
$$sH = \frac{eH\tau}{m} = \frac{eHt}{m\gamma} = \omega t, \quad \frac{mv_0\gamma}{eH} = \frac{p_0}{eH} = r, \tag{33}$$

where $u_x(0), u_y(0)$ are the initial component of the transverse 4-velocity, ω is the Larmour frequency. Substituting all this into Eq.(32) one finds the trajectory of particle

$$x^1 = x = r(\sin\omega t - \beta) + x(0),$$
$$x^2 = y = r(\cos\omega t - \beta) + y(0) \qquad (34)$$

So in the constant magnetic field the charged particle moves along a helix having its axis along the magnetic field and radius given in Eq.(33). In the particular case $u_z(0) = 0$, that is the particle have no velocity component along the field, it moves along a circle in the plane transverse to the field.

Solution in electric field

In special frame we are using the electric field is directed along axis 3(z). From Eq.(27) one has the equation for u

$$u = (C^2 \cosh sE + C \sinh sE - B^2)u(0) \qquad (35)$$

The initial condition is: $u^1 = u^3 = 0, u^2 = u_0$ so that $u^0(0) = \sqrt{1 + u_0^2}$ (see Eq.(14)).

From Eq.(35) one has for components of the 4-velocity:

$$u^0 = \cosh sEu^0(0), \quad u^3 = -\sinh sEu^0(0), \quad u^2 = u_0, \quad sE = \frac{eEt}{m\gamma}. \tag{36}$$

In an electric field the solution Eq.(29) (taking into account Eq.(31)and the chosen initial condition) is

$$x = \frac{m}{eE}(C^2 \sinh sE + C(\cosh sE - 1) - B^2 sE)u(0), \tag{37}$$

or

$$x^0(s) = \frac{m}{eE}\sinh sEu^0(0), \quad x^3(s) = z = \frac{m}{eE}(\cosh sE - 1)u^0(0),$$
$$x^2(s) = y = -\frac{m}{e}u_0 s. \tag{38}$$

From the last expression one has $s = -ey/p_0$ (here $p_0 = mu_0$ is the initial (transverse) momentum), so that $sE = eEy/p_0$. substituting this result into expression for z one finds the expression for the trajectory

$$z = \frac{\mathcal{E}_0}{eE}\left[\cosh\frac{eEy}{p_0} - 1\right], \tag{39}$$

where $\mathcal{E}_0 = mu_0(0)$ is the initial energy of a particle.

Thus in a constant electric field a charge moves along a catenary curve.

Vacuum Electron Loops

Diagram approach

There is a class of diagrams which determine the amplitude of the transition of the vacuum into a vacuum. In the presence of an external field, these diagrams describe processes that are in principle observable, for example pair production by an external electric field.

We consider, for the sake of arguments, vacuum electron loops. Let $L^{(n)}$ be the amplitude describing a closed loop of free particles interacting with an external field n times. The explicit form of the amplitude in the coordinate representation was already discussed by Feynman

$$L^{(n)} = \frac{ie^n}{n(2\pi)^4} \int \int .... \int d^4x_1 d^4x_2 .... d^4x_n \mathrm{Tr}[G(x_1, x_2)\hat{A}^F(x_2)G(x_2, x_3)$$
$$\times \hat{A}^F(x_3)....G(x_{n-1}, x_n)\hat{A}^F(x_n)G(x_n, x_1)\hat{A}^F(x_1)] \quad (40)$$

It is convenient to use the following representation of $G(x_i, x_j)$

$$G(x_i, x_j) = \left\langle x_i \left| \frac{1}{\hat{p} - m} \right| x_j \right\rangle, \quad (41)$$

where $\hat{p} = \gamma^\mu p_\mu, p_\mu = i(\hbar)\partial_\mu$. Here $|x >$ is the eigenvector of the coordinate operator $X|x >= x|x >$ normalized in such a way that

$$< x|x' >= \delta(x - x'), \quad \int d^4x|x >< x| = 1. \quad (42)$$

Using the self-adjoint character of the operator p_μ we have $L^{(n)}$ in a form convenient for analysis

$$L^{(n)} = \frac{i}{n(2\pi)^4} \int d^4x \mathrm{Tr} \left\langle x \left| \left(\frac{1}{\hat{p} - m + i\epsilon} e\hat{A}^F \right)^n \right| x \right\rangle \tag{43}$$

The contribution of electron loops with any number of interaction with an external field $L = \sum_n L^{(n)}$ contains the sum

$$\sum_n \frac{1}{n} \left(\frac{1}{\hat{p} - m + i\epsilon} e\hat{A}^F \right)^n = -\ln \left(1 - \frac{1}{\hat{p} - m + i\epsilon} e\hat{A}^F \right) = -\ln \left(\frac{\hat{P} - m}{\hat{p} - m} \right), \tag{44}$$

where $P_\mu = p_\mu - eA_\mu^F$. It is convenient to represent the logarithm of the last operator expression in the form of a Frullani integral:

$$\ln\left(\frac{\hat{P}-m}{\hat{p}-m}\right) = -\int\limits_0^\infty \frac{ds}{s}[\exp(is(\hat{P}-m+i\epsilon)) - \exp(is(\hat{p}-m+i\epsilon))]. \quad (45)$$

The last term of the expression in the right-hand side corresponds to substraction at the point $F = 0(A = 0)$. We shall not write it out for the time being, and take it into account in the final expression. Thus, the problem has been reduced to a determination of the quantity

$$L = \int d^4x\mathcal{L}, \quad (46)$$

where

$$\mathcal{L} = \frac{i}{(2\pi)^4} \mathrm{Tr} \int\limits_0^\infty \frac{ds}{s} \left\langle 0 \left| \exp(is(\hat{P} - m + i\epsilon)) \right| 0 \right\rangle . \tag{47}$$

We have used here the translation invariance in a homogeneous field: the average $\langle 0 \,|....|\, 0 \rangle$ does not depend on the coordinate.

The fact that the trace of an odd numbers of γ matrices vanishes makes it possible to carry out the transformation

$$\mathrm{Tr}\left[\ln\left(\frac{\hat{P} - m}{\hat{p} - m}\right)\right] \to \frac{1}{2}\mathrm{Tr}\left[\ln\left(\frac{\hat{P}^2 - m^2}{\hat{p}^2 - m^2}\right)\right], \tag{48}$$

which yields a representation that is more convenient for the calculation

$$\mathcal{L} = \frac{i}{2(2\pi)^4} \mathrm{Tr} \int\limits_0^\infty \frac{ds}{s} \left\langle 0 \left| \exp[is(\hat{P}^2 - m^2)] \right| 0 \right\rangle . \tag{49}$$

Let us take into account that

$$\hat{P}^2 = P^2 - \frac{e\sigma^{\mu\nu}}{2} F_{\mu\nu}, \quad \sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{50}$$

Solution of operator equation

So, we have to calculate

$$N(s) = \left\langle 0 \left| \exp[isP^2] \right| 0 \right\rangle \tag{51}$$

We introduce the function $N(\alpha s)$ and differentiate it with respect to α:

$$\frac{dN(\alpha s)}{d\alpha} = is \left\langle 0 \left| P^2 \exp[i\alpha s P^2] \right| 0 \right\rangle \tag{52}$$

The operator $\exp[isP^2]$ is the displacement operator in "time" s so that

$$X(s) = \exp[isP^2] X \exp[-isP^2], \quad P(s) = \exp[isP^2] P \exp[-isP^2] \tag{53}$$

Differentiating these operators with respect to s

$$\frac{dX(s)}{ds} = i \exp[isP^2](P^2 X_\nu - X_\nu P^2) \exp[-isP^2],$$
$$\frac{dP(s)}{ds} = i \exp[isP^2](P^2 P_\nu - P_\nu P^2) \exp[-isP^2] \tag{54}$$

and taking into account commutators:

$$[P_\mu, X_\nu] = i\delta_{\mu\nu}, \quad [P^2, X_\nu] = 2iP_\nu, \quad [P_\mu, P_\nu] = -ieF_{\mu\nu}, \quad [P^2, P_\nu] = 2ieF_{\mu\nu}P^\nu \tag{55}$$

we obtain (in matrix form)

$$\frac{dP(s)}{ds} = -2eFP, \tag{56}$$

so that

$$P(s) = \exp(-2eFs)P(0) \tag{57}$$

For the coordinate operator one has

$$\frac{dX(s)}{ds} = -2P(s) = -2\exp(-2eFs)P(0) \tag{58}$$

The solution of this equation is

$$X(s) - X(0) = U(s)P(0), \quad U(s) = \frac{\exp(-2eFs) - 1}{eF} \tag{59}$$

Here appears the very important matrix $U(s)$. From Eqs.(58)-(59) one has

$$\begin{aligned} P(s) &= \exp(-2eFs)U^{-1}[X(s) - X(0)] \\ P^T(s) &= [X(s) - X(0)](U^{-1})^T \exp(2eFs), \quad F^T = -F(F_{\mu\nu} = -F_{\nu\mu}) \end{aligned} \tag{60}$$

As a result we find

$$P^2(s) = [X(s) - X(0)](U^{-1})^T U^{-1}[X(s) - X(0)] \tag{61}$$

Taking into account that (see Eq.(59))

$$\frac{dU(\alpha s)}{d\alpha} = -2s\exp(-2eFs\alpha),$$
$$\frac{1}{2}\frac{dU(\alpha s)}{d\alpha}\frac{1}{U(s\alpha)} = \frac{seF}{\exp(2eFs\alpha) - 1} = -s(U^{-1}(s\alpha))^T \tag{62}$$

The equation Eq.(52) reduced to

$$\frac{dN(s\alpha)}{d\alpha} = B(\alpha)N(s\alpha), \tag{63}$$

where

$$B(\alpha) = -s\mathrm{Tr}[(U^{-1}(s\alpha))^T], \tag{64}$$

so the solution of Eq.(63) is

$$N(s) = C\exp(\int B(\alpha)d\alpha). \tag{65}$$

Here C is the constant which can be calculated in the limit $F \to 0, C = -i\pi^2$. and we have to calculate Tr

$$s\mathrm{Tr}\int (U^{-1}(s\alpha))^T d\alpha = -\mathrm{Tr}\int \frac{1}{2}\frac{dU(\alpha s)}{d\alpha}\frac{1}{U(s\alpha)}d\alpha = -\frac{1}{2}\mathrm{Tr}\ln U(s). \tag{66}$$

Final reduction

so the solution is

$$N = C \exp\left(-\frac{1}{2}\mathrm{Tr}\ln\left(\frac{-U(s}{2}\right)\right) = \frac{C}{\sqrt{\exp(\mathrm{Tr}\ln(-U(s)/2))}} = \frac{C}{\sqrt{\det(-\mathrm{U(s)}/2)}}, \quad (67)$$

where the known mathematical formula is used

$$\exp(\mathrm{Tr}\ln D) = \det D, \quad (68)$$

here D is the matrix.

This determinant can be calculated using found above representation

$$U = \frac{\exp(-2esF) - 1}{eF} = a_1 C^2 + a_3 C + a_2 B^2 + a_4 B, \quad (69)$$

where

$$a_1 = \frac{\sinh(2sE)}{eE}, \quad a_2 = \frac{\sin(2sH)}{eH}, \quad a_3 = \frac{\cosh(2sE) - 1}{eE}, a_4 = \frac{1 - \cos(2sH)}{eH}. \tag{70}$$

So

$$\det U = \det \begin{pmatrix} a_1 & 0 & 0 & a_3 \\ 0 & a_2 & -a_4 & 0 \\ 0 & a_4 & a_2 & 0 \\ -a_3 & 0 & 0 & -a_1 \end{pmatrix} = (a_2^2 + a_4^2)(a_3^2 - a_1^2) = -\frac{16\sin^2(eHs)\sinh^2(eEs)}{e^4H^2E^2} \tag{71}$$

The final result is

$$N = \frac{C}{\sqrt{\exp(\mathrm{Tr}\ln(-U(s)/2))}} = -i\pi^2\Phi(s), \quad \Phi(s) = \frac{e^2EH}{\sin(eHs)\sinh(eEs)} \tag{72}$$

The last part of the calculation:the term with γ matrices

$$\exp(-ia\frac{e\sigma^{\mu\nu}}{2}F_{\mu\nu}) = \exp(-ia\mathbf{\Sigma H})\exp(a\boldsymbol{\alpha}\mathbf{E}), \tag{73}$$

where $\mathbf{\Sigma}(\Sigma_1, \Sigma_2, \Sigma_3)$ and $\boldsymbol{\alpha}(\alpha_1, \alpha_2, \alpha_3)$ are the standard matrices of the Dirac theory. Taking into account that

$$\begin{aligned} \exp(-ia\mathbf{\Sigma H}) &= \cos aH - \frac{i\mathbf{\Sigma H}}{H}\sin aH, \\ \exp(\boldsymbol{\alpha}\mathbf{E}) &= \cosh aE + \frac{\boldsymbol{\alpha}\mathbf{E}}{E}\sinh aE \end{aligned} \tag{74}$$

and $\mathrm{Tr}\mathbf{\Sigma}$ $=\mathrm{Tr}\boldsymbol{\alpha}$=0 we find

$$\mathrm{Tr}\exp(-is\frac{e\sigma^{\mu\nu}}{2}F_{\mu\nu}) = 4\cosh seE\cos seH \tag{75}$$

So

$$\mathrm{Tr}\left\langle 0 \left| \exp[is\hat{P}^2] \right| 0 \right\rangle = -4i\pi^2 e^2 HE \cot(seH)\coth(seE) \tag{76}$$

Substituting this result to Eq.(57) end performing subtraction of the two first terms of expansion of found expression at $H, E \to 0$ (the first corresponds to the subtraction term as $F \to 0$, the second appears in renormalization of charge) we have for effective Lagrangian in a constant electromagnetic field

$$\mathcal{L} = \frac{1}{8\pi^2} \int\limits_0^\infty \frac{ds}{s^3} \exp[-is(m^2 - i\epsilon)] \Big[s^2 e^2 EH \cot(seH)\coth(seE)$$

$$-1 - \frac{s^2 e^2}{3}(E^2 - H^2) \Big] \tag{77}$$

Pair creation in an electric field

This effective lagrangian was first derived by Heisenberg and Euler in 1935 with the aid of different approach. In accordance with meaning of the function $\mathcal{L}$ the probability of pair production in a unit 4-volume is 2 Im $\mathcal{L}$. The last quantity can be calculated by rotating the contour of integration in Eq.(77) through $-\pi/2$ and evaluating the integrals with the aid of residue theory. At mentioned rotation $s \to -is$

$$\exp[-ism^2] \to \exp[-sm^2], \quad \cot(seH) \to \coth(seH), \quad \coth(seE) \to \cot(seE) \tag{78}$$

The function $\cot(seE) = 0$ at $s = n\pi/(eE)$ and one obtains pair creation probability as a sum over residue in this poles:

$$W = \frac{e^2 EH}{4\pi^2} \sum_n \frac{1}{n} \coth\left(\frac{n\pi H}{E}\right) \exp\left(-\frac{n\pi E_0}{E}\right), \tag{79}$$

where $E_0 = m^2/e = (m^2c^3/(e\hbar)) = 1.32 \cdot 10^{16} V/cm$.

In the pure electric field $H = 0$ one has

$$W = \frac{e^2 E^2}{4\pi^3} \sum_n \frac{1}{n^2} \exp\left(-\frac{n\pi E_0}{E}\right) , \tag{80}$$

Physics of heavy atoms and positron creation

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Solution of Dirac equation at High Z

The Dirac equation in Coulomb field $U = -Ze^2/r$ can be solved exactly and the energy levels are

$$\frac{\epsilon}{m} = \left[1 + \frac{\xi^2}{n - |\kappa| + \sqrt{\kappa^2 - \xi^2}}\right]^{-1/2}, \tag{1}$$

where $\xi = Z\alpha = Z/137, n$ is the principal quantum number $n = 1, 2, 3...$ and $\kappa = \mp(j + 1/2)$ for $j = l \pm 1/2$, j is the total momentum of the electron. For the ground state $(n = \kappa = 1, j = 1/2)$ one has

$$\frac{\epsilon}{m} = \sqrt{1 - \xi^2} \tag{2}$$

In the limit $\xi \ll 1$ one has

$$\frac{\epsilon}{m} - 1 = -\frac{(Z\alpha)^2}{2}. \tag{3}$$

It is seen that atomic units are used, for hydrogen $Z = 1$ the ground state is $\epsilon - m = -m\alpha^2/2 \simeq 13.6$ eV.

The energy levels Eq.(1) are obtained for the point source. Actually any nucleus has finite size. This means that one can't make a choice of boundary condition at zero. Technically, this means that the problem must be solved with a potential that is cut off at $r < R$. As a result Eq.(2) is not applicable at $\xi \to 1$. The solution exists from $\xi = 0, \epsilon/m = 1$ to $\xi = \xi_c, \epsilon/m = -1$. The estimates show that $\xi_c = 1.25 (Z_c = 170)$ (see Fig.1).

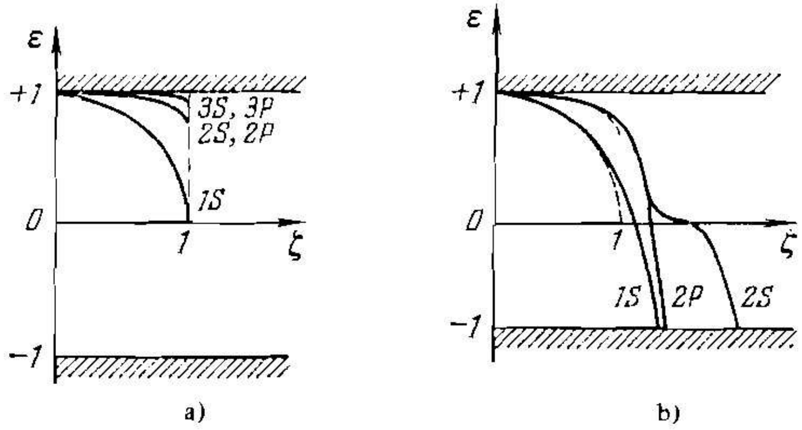

Figure 1: The energies of the lowest levels with $j = 1/2(\xi = Z/137)$:(a) for point-charge potential $V(r) = -Ze^2/r$: (b) with allowance for the finite dimensions of the nucleus.

It is necessary to take into account that we Dirac equation (two equation for two-component spinors), which can be reduced to a singe second order equation having formally the same form as the nonrelativistic Schredinger equation with certain effective potential U. The effective potential is shown in Fig.2

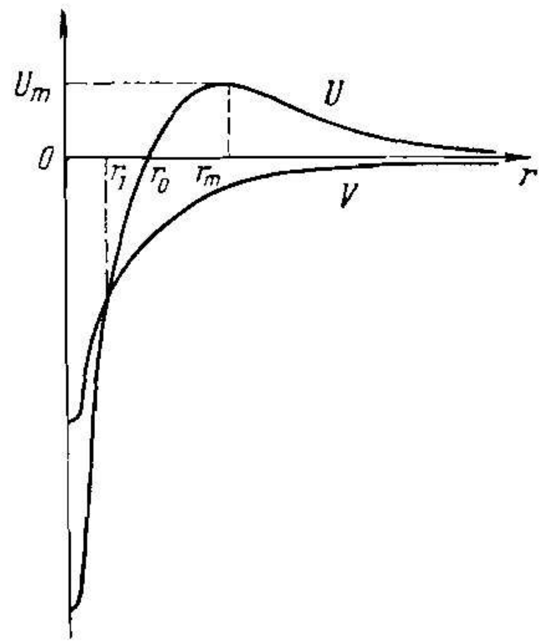

Figure 2: The potential $V(r)$ and the effective potential $U(r)$ for the Coulomb problem at $z \to Z_c$ and $\epsilon \to -m$

Dirac equation in Coulomb field

We expand the field operator $\psi(x)$ in terms of exact solutions of the Dirac equation in the Coulomb field of nucleus Z:

$$\psi(x) = \sum_{(+)} a_n \psi_n(x) + \sum_{(-)} b_n^+ \bar{\psi}_n(x) \tag{4}$$

Here $(+)$ denotes the sum over electron states (i.e. sum over states of discrete spectrum $-1 < \epsilon_n/m < 1$ and integral over the upper continuum $\epsilon/m > 1$), and $(-)$ denotes the sum over positron states (lower continuum $\epsilon/m < -1$). All the level with energy $-1 < \epsilon_n/m < 1$ are classified as electron states, for when Z is adiabatically decreased they returned to the upper continuum. For particles obeying the Dirac equation, there are no levels that go from the lower continuum (i.e. there are no bound states for the antiparticle at a given sign of the potential $V(r)$).

Let us consider the Heisenberg current operator

$$j_\mu(x) = -\frac{e}{2}[\bar{\psi}(x), \gamma_\mu \psi(x)], \tag{5}$$

where $-e$ is the charge of electron ($e > 0$). The mean value of $j_0(\mathbf{x})$ in an arbitrary state φ determines the charge density

$$\varrho(\mathbf{r}) = < \varphi | j_0(\mathbf{x}) | \varphi > = -e \sum_{(+)} N_+ |\psi(\mathbf{r})|^2 + e \sum_{(-)} N_- |\psi(\mathbf{r})|^2$$
$$+ \frac{e}{2} \left[\sum_{(+)} |\psi(\mathbf{r})|^2 - \sum_{(-)} |\psi(\mathbf{r})|^2 \right] , \tag{6}$$

where $N_+ = a_n^+ a_n$ and $N_- = b_n^+ b_n$ are the occupation numbers (the spacial part of the current $\mathbf{j}(\mathbf{x})$ gives zero when averaged over the stationary state φ). The last term in Eq.(6) is obviously the charge density of the vacuum

$$\varrho_{vac}(\mathbf{r}) = \frac{e}{2}\left[\sum_{(+)} |\psi(\mathbf{r})|^2 - \sum_{(-)} |\psi(\mathbf{r})|^2\right] . \tag{7}$$

We called here the state with $N_+ = N_- = 0$ the vacuum state (in the language of the initial Dirac theory this corresponds to a completely filled lower continuum). It is convenient to retain this definition, which is obvious when $Z < Z_c$, also in the region of supercritical $Z > Z_c$.

In the field of nucleus Z, the wave function ceases to be symmetrical with respect to the sign of ϵ. The resultant distortion $|\psi_\epsilon(\mathbf{r})|^2 - |\psi_{-\epsilon}(\mathbf{r})|^2$ determines the charge density induced in the vacuum (i.e. polarization of the vacuum). So long as $Z < Z_c$, the total charge of the vacuum remains equal to zero

$$Q_{vac} = \int \varrho_{vac}(\mathbf{r}) d^3r = 0 \tag{8}$$

In the region $\xi \ll 1$ Eq.(7) leads to the well-known Uehling potential

$$\varphi(r) = \varphi_0 + \delta\varphi = Ze\left[\frac{1}{r} + \frac{2\alpha}{3\pi}\int\limits_{2m}^{\infty} d\mu\sigma(\mu)\frac{\exp(-\mu r)}{r}\right], \tag{9}$$

where

$$\sigma(\mu) = \frac{\sqrt{\mu^2 - 4m^2}}{\mu^2}\left(1 + \frac{2m^2}{\mu^2}\right) \tag{10}$$

In the limiting cases this formula gives

$$\varphi(r) = \frac{Ze}{r}\left[1 + \frac{\alpha}{4\sqrt{\pi}}\frac{\exp(-2mr)}{(mr)^{3/2}}\right], \quad r \gg \frac{1}{m},$$
$$\varphi(r) = \frac{Ze}{r}\left[1 + \frac{2\alpha}{3\pi}\left(\ln\frac{1}{mr} - C - \frac{5}{6}\right)\right], \quad r \ll \frac{1}{m}, \tag{11}$$

where $C = 0.577\ldots$ we see that the polarization of the vacuum alters the Coulomb field of the point charge in the region $r \leq 1/m(\hbar/mc)$.

Taking into account the equation

$$(\Delta - \mu^2)\frac{\exp(-\mu r)}{r} = -4\pi\delta(\mathbf{r}), \tag{12}$$

we obtain the charge density corresponding to Eq.(9)

$$\varrho(\mathbf{r}) = -\frac{\Delta\varphi}{4\pi} = Ze\left[\delta(\mathbf{r}) + \frac{2\alpha}{3\pi}\int\limits_{2m}^{\infty} d\mu\sigma(\mu)\left(\delta(\mathbf{r}) - \frac{\mu^2\exp(-\mu r)}{4\pi r}\right)\right] \tag{13}$$

The charge distribution has the following form:a point-like positive charge $(1+\gamma Ze)$ at the center $\mathbf{r}=0$ and a cloud of negative charges (adding to $-Ze$) smear out a distance $r \sim 1/m$ from nucleus. The total charge of the system is Ze, as follows formally from identity

$$\int d^3r\left[\delta(\mathbf{r}) - \frac{\mu^2\exp(-\mu r)}{4\pi r}\right] = 1 - 1 = 0. \tag{14}$$

The region $\xi = Z\alpha > 1$

Let us consider situation when charge of nucleus charge Z passes through the critical value Z_c. We present radial component of the solution of the Dirac equation in Coulomb field ψ in the form

$$\psi(r) = r \begin{pmatrix} g \\ f \end{pmatrix} . \tag{15}$$

Assuming that density

$$\varrho_\epsilon = |g_\epsilon(r)|^2 + |f_\epsilon(r)|^2, \tag{16}$$

where $g_\epsilon(r)$ and $f_\epsilon(r)$ are the radial functions for the energy ϵ, we have

$$\varrho_{vac} = \frac{e}{2} \left[\sum_{-1<\epsilon/m<1} \varrho_n + \int_0^\infty dp[\varrho_\epsilon - \varrho_{-\epsilon}] \right] , \tag{17}$$

It will be convenient to denote the quantities pertaining to the cases $Z < Z_c$ and $Z > Z_c$ by the indices - and $+$, respectively. At $Z \sim Z_c$, abrupt changes are experienced only the wave functions with energy ϵ close to $-m$, namely:

1. At $Z \sim Z_c$ the sum over states of discrete spectrum contains the term ϱ_m corresponding to the level $1S$ and at $Z > Z_c$ this term vanishes from from the complete set of the single-particle functions.
2. On the other hand, when $Z > Z_c$ the lower-continuum functions with angular momentum $j = 1/2$ experience strong perturbations (they increased sharply at small r). This perturbation can be described by the formula

$$\varrho_\epsilon^+ - \varrho_\epsilon^- \simeq \Delta(p)\varrho_c(r), \tag{18}$$

where

$$\Delta(p) = \frac{\gamma}{2\pi[(p-p_0)^2 + \gamma^2/4]}, \quad p = \sqrt{\epsilon^2 - m^2}, \tag{19}$$

$\varrho_c(r) = |g_{\epsilon_C}(r)|^2 + |f_{\epsilon_C}(r)|^2$ is the square of radial wave function at $Z = Z_c$ and $\epsilon = -m$.

For the remaining states, the point $Z = Z_c$ is not distinguished in any way, and these states produced in ϱ_{vac} a background that varies with $Z - Z_c$.

Thus, the change of ϱ_{vac} on going through Z_c is equal to

$$\varrho^{+}_{vac}(r) - \varrho^{-}_{vac}(r) = -\frac{e}{2}\left[2\varrho_c(r) + 2\int\limits_0^\infty dp\Delta(p)\varrho_c(r)\right] = -2e\varrho_c(r), \qquad (20)$$

where the factor 2 reflects the double degeneracy with respect of spin projection.

When $Z > Z_c$, the vacuum becomes charged. the total charge of vacuum is equal to $-2e$. For the external observer the effective charge of nucleus Z decreases to $Z - 2$. If we have at $Z < Z_c$ an uncharged vacuum (a bare nucleus whose $1S$ level is not occupied by electrons), then spontaneous emission of two positrons occurs and the vacuum acquires two units of negative charge of two units of negative charge in accord with charge conservation law.

Let us discuss what means "charged vacuum". In the both cases $Z < Z_c$ and $Z > Z_c$ we define as the vacuum the lowest energy state in the field of nucleus with charge Z. As long as $Z < Z_c$, it corresponds to a bare nucleus with unfilled shells. On going through $Z = Z_c$, the bare nucleus becomes metastable and realignment of the vacuum takes place (emission of two positrons).

So, if it is possible to combine two bare uranium nuclei and produce a nucleus with supercritical $Z = 184 > Z_c$ then one can expect spontaneous emission of two positrons.

Cross section of positron production

We consider collision of two heavy nucleus. Let us introduce notations: m_N is the nucleon mass, A is atomic number, E is the kinetic energy of colliding nucleus, E_t is the threshold of spontaneous positron creation, R is the distance between nucleus in units $\hbar/mc = 386f$, at $R < R_c$ the lowest electron level is passed into the lower continuum on the energy level $\epsilon = \epsilon_0 + i\gamma/2$.

Since the ratio $m/Am_N \sim 10^{-6}$, collision of nucleus may be considered in frame of classical theory, their relative velocity $\sqrt{2E/(Am_N)} \sim 10^{-2}$ for energy $E \sim 10$ MeV, so it is much less than velocity of electron in S state at $Z\alpha \sim 1$. Because of this the adiabatic approximation is valid and the cross section of the process is expressed in terms of γ. For simplicity the symmetric case $Z_1 = Z_2 = Z$

is considered. The important parameter R_c is

$$R_c = \frac{c}{m}\exp\left(-\frac{\pi}{\sqrt{\zeta^2-1}}\right)), \quad c \simeq 1 \tag{21}$$

where $\zeta = 2Z\alpha$. The width of the energy level is narrow:$\gamma \ll \epsilon_0$. One can introduce the variable

$$y = -\frac{\epsilon_0}{k}, \quad 1 < y < \infty, \tag{22}$$

where $k = (y^2-1)^{-1/2}$ and $v = 1/y$ are the momentum and velocity of outgoing positrons. Values of ϵ_0 and γ are

$$\epsilon_0 = -\sqrt{1+k^2} = -\frac{y}{\sqrt{y^2-1}}$$

$$\gamma(y) = 2\pi \left[(\exp(2\pi y) - 1)[(y^2-1)^{3/2}\mathrm{Im}\psi'(iy) + y(y^2-1)^{1/2} + \frac{1}{2}\left(1 - \frac{1}{y^2}\right)] \right]^{-1} \quad (23)$$

where $\psi(y)$ is the logarithmic derivative of the Γ-function.

The approximate cross section of the positron creation can be written as

$$\sigma(E, Z) = \sigma_0(Z) f(\eta), \quad (24)$$

where $\eta = E/E_t = R_c/R_0, R_0$ is the distance of minimal closing in of colliding nucleuses. The first multiplier $\sigma_0(Z)$ is defined by the critical distance R_c, for example for uranium $\sigma_0(Z) \simeq 3 \cdot 10^{-23} cm^2$. The function $f(\eta)$ defines the dependence of cross section on energy

$$f(\eta) = \frac{4\pi}{\sqrt{\eta}} \int\limits_{1/\eta}^{1} \gamma(x) x^{3/2} \sqrt{x - \frac{1}{\eta}} dx, \tag{25}$$

where $\gamma(x)$ is defined in Eq.(23). Near threshold of positron creation

$$f(\eta) = f_0(\eta - 1)^{9/4} \exp[-b(\eta - 1)^{-1/2})], \tag{26}$$

where f_0, b are constant, in the limit $\delta \to 0$ one has $f_0 = 8.7, b = 5.73$.

Near to threshold the cross section Eq.(24) is exponentially small: this is the consequence of the Coulomb barrier for slow positrons. However the function

$f(\eta)$ increases fast with energy grow. At $\eta \geq 2$ the function $f(\eta) \sim 10^{-3}$ and at further η increase its dependence on energy is very slow. However background effects are increasing. So, the region $E \simeq 2E_t$ may be optimal.

Probability of e^+ creation at nucleus scattering at given angle

The angular distribution of positrons is

$$\frac{d\sigma}{d\Omega} = \sigma F(\vartheta, \eta), \tag{27}$$

where σ is the total cross section Eq.(24), $\eta = E/E_t > 1$,

$$F(\vartheta,\eta) = \frac{\eta F_1(\vartheta,\eta)}{32\pi F_2(\eta)\sin^4(\vartheta/2)}, \quad F_1(\vartheta,\eta) = \int\limits_{\rho_+}^{1} \gamma(x) \frac{x dx}{\sqrt{(x-\rho_+)(x+\rho_-)}},$$

$$F_2(\eta) = \int\limits_{1}^{\eta} \gamma(x/\eta) x^{3/2} \sqrt{x-1} dx, \quad \rho_\pm = \frac{\pm 1 + 1/\sin(\vartheta/2)}{2\eta}, \tag{28}$$

where ϑ is the scattering angle in c.m.f. The function F is normalized:

$$\int\limits_{\vartheta_1}^{\pi} F(\vartheta,\eta)\sin\vartheta d\vartheta = \frac{1}{2\pi}. \tag{29}$$

Here ϑ_1 is the minimal scattering angle when positron creation is possible:

$$\vartheta_1 = \pi - 4\sqrt{\eta - 1}, \quad \eta \to 1; \quad \vartheta_1 = \frac{1}{\eta}, \quad \eta \gg 1 \tag{30}$$

At $\vartheta = \vartheta_1$ one has $\rho_+ = 1$ and $F_1 = F = 0$. At the threshold $E \to E_t$ the function $F(\vartheta, \eta)$ looks as narrow peak near $\vartheta = \pi$, i.e. the positrons created at back-scattering only. With E increase the angle interval enlarges fast. At $\eta > 2$ the maximum of $d\sigma/d\Omega$ displaces to angle smaller than π. This is due to the Rutherford factor $1/\sin^4 \vartheta/2$ in Eq.(28).

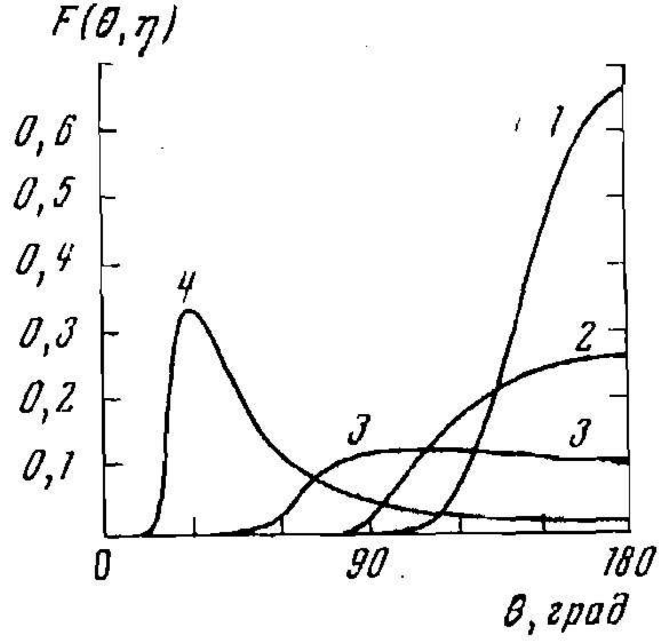

Figure 3: The relative probability of positron creation at projectile scattering at angle ϑ. The curves 1-4 are respectively for $\eta = E/E_t$=1.25; 1.5; 2 and 5.

The energy spectrum of created positrons can be presented in a form

$$\frac{d\sigma}{dT} = \frac{2\pi C}{\exp(2\pi\zeta/v) - 1} u^{5/2}\sqrt{u-1}. \quad (31)$$

Here T is the kinetic energy of positron, which is in interval $0 \leq T \leq T_{max}$

$$T_{max} = |\epsilon_0(1/\eta)| - m, \quad (32)$$

v is the positron velocity, $T = m(1/\sqrt{1-v^2} - 1)$. The variable u is connected with T by the equation

$$\epsilon_0(u/\eta) = -(T+1), \quad 1 < u < \eta. \quad (33)$$

The constant C is defined from normalization. For notations see Eqs.(21)-(23). Near the threshold all positrons are non-relativistic and the cross section Eq.(31)

can be written as

$$\frac{d\sigma}{dT} = C\sqrt{T_{max} - T}\exp(-a/\sqrt{T}). \tag{34}$$

The positron spectra are shown in Fig.4.

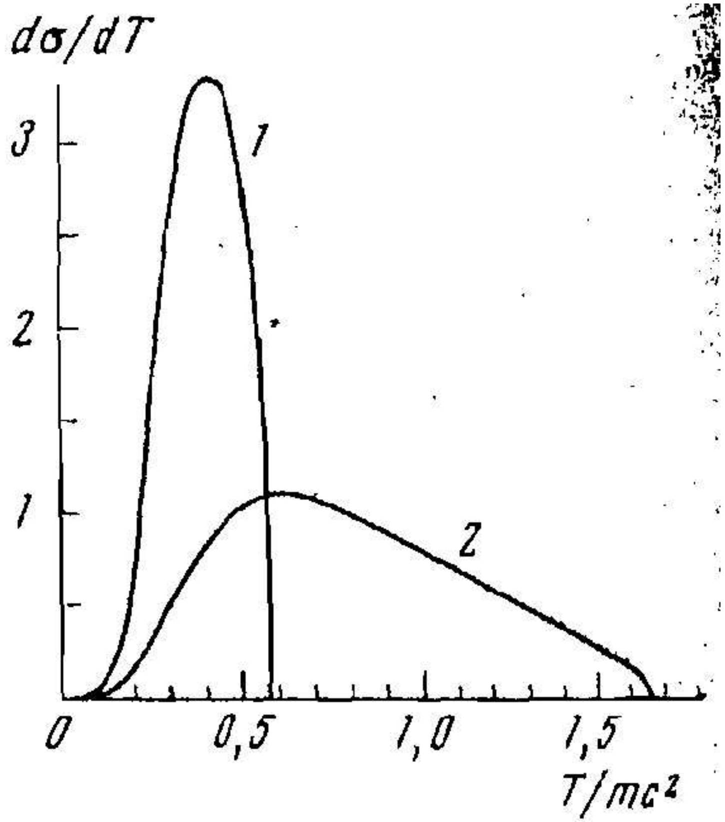

Figure 4: The positron spectrum for $\eta = 2$ (the curve 1) and for $\eta = 4$ (the curve 2)

Experimental Study of Positron Production

There is a series different mechanisms which produced positrons at heavy nucleus collision:

1. Production electron-positron pair in collision of two charged particles according to QED (Landau-Lifshitz mechanism).

2. Internal conversion of γ rays from exited nuclear states.

3. External conversion of γ rays in a target.

4. Spontaneous positron creation at collision of two nucleus $(Z_1 + Z_2 > Z_c)$

Pioneering experiment of this type were carried out at GSI (Darmstadt). The production of positrons in collision of high-Z nuclei was observed with a continuous spectrum of energies centered at approximately 400 keV and width of about 1 MeV.

Some results are shown in Fig.5,6

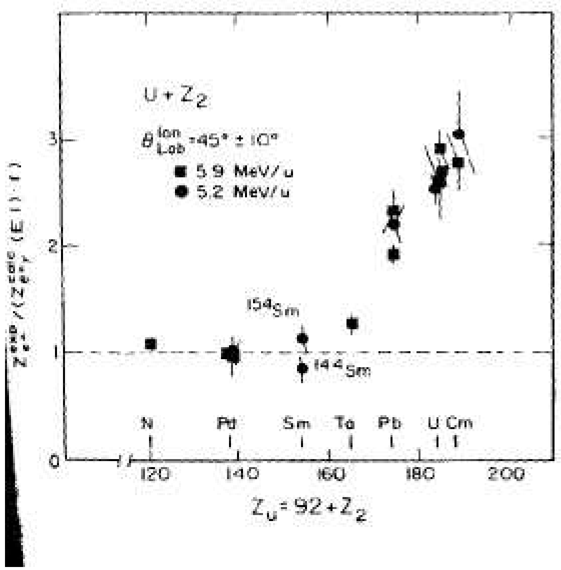

Figure 5: Positron yield in Uranium + Z_2 compared with yield expected from conversion of the measured γ-ray spectra. The steep rise beyond $Z_1 + Z_2 = 160$ may indicate spontaneous positron creation. (Fig.13.24 from GMR)

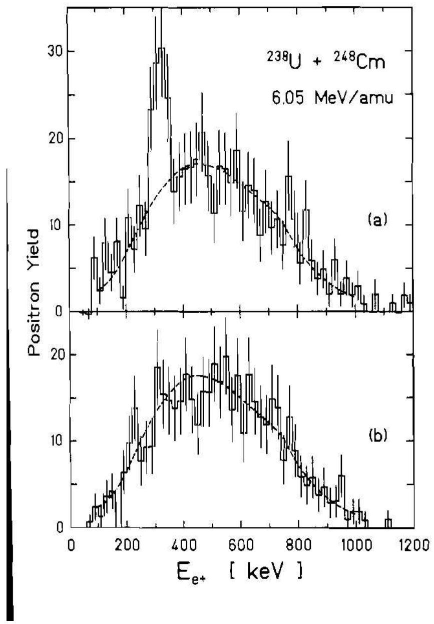

Figure 6: Positron energy spectra observed for $\mathrm{U}^{238} + \mathrm{Cm}^{248}$ collision at a projectile energy 6.05 MeV/u. Kinematics selection overlap preferentially with elastic scattering angles of (a) $100° < \vartheta_{cm} < 130°$ and (b) $50° < \vartheta_{cm} < 80°$. (Fig.13.33 from GMR)

Conclusion made in GMR (W.Greiner, B.Mueller, J.Rafelski "Quantum Electrodynamics in Strong Fields", Springer, 1985):

1. ......

2. A reaction model has been constructed that consistently explains the observed peak in the U+Cm systems as a *spontaneous positron emission* line from the K-shell of a metastable giant nuclear molecular system.

3. ......

Further analysis of these data showed that observed positrons originated mainly from two sources: internal pair decay of exited states in the colliding nuclei and from pair produced by strong transient electromagnetic fields presented in collisions. The sought-for "spontaneous positron production" could not be isolated from the transient "dynamic" process.

Interest to these studies was heightened when unexpected narrow structures were observed in the measured positron spectra. similar structures were seen in a variety of collision systems. The width of these features corresponded to the value expected from monoenergetic source moving with approximately center-of-mass velocity of the collision system. These observation led to a number of theoretical speculations, one of which was that the origin of the narrow positron lines was the two-body decay of a slow moving neutral object into a positron-electron pair. The existence of such an object would require new physics, such as new light neutral elementary particle, or a novel narrow state of positron-electron. Such possibility were, however, severely constrained by other results.

The suggestion of possible new physics prompted a new generation of heavy-ion scattering experiment to detect positrons and electrons in coincidence. Very sharp sum-energy peaks were found in coincidence spectra.

Last results

In the quite recent analysis I.Ahmad et al, Phys.Rev.C,**60** 064601 (1999) "Electron-positron pair produced in heavy-ion collisions"

TABLE I. Summary of experimental characteristics of previously reported $e^+ - e^-$ coincidence lines.

System	$e^+ - e^-$ sum energy (keV)	Line width (keV)	Beam energy (MeV/nucleon)	Energy loss in target (MeV/nucleon)	Cross section (μb/sr) (iso)[a]	Cross section (μb/sr) (bb)[b]	Original reference
$^{238}U + ^{232}Th$	608±8	25±3	5.86–5.90	0.07	2.7±0.6	1.1±0.3	[20,21]
$^{238}U + ^{232}Th$	760±20	≤80	5.83	0.07	–	–	[18]
$^{238}U + ^{232}Th$	809±8	40±4	5.87–5.90	0.07	3.1±0.7	1.3±0.3	[20,21]
$^{238}U + ^{181}Ta$	625±8	20±3	6.24–6.38	0.10	3.2±0.8	1.3±0.3	[20,21]
$^{238}U + ^{181}Ta$	748±8	33±5	5.93–6.13	0.10	5.7±1.3	2.3±0.5	[20,21]
$^{238}U + ^{181}Ta$	805±8	27±3	6.24–6.38	0.10	3.3±0.8	1.4±0.4	[20,21]
$^{238}U + ^{181}Ta$	≈635	≈30	6.30	0.24	0.5±0.1	–	[22]

[a]Cross section $d\sigma_{line}/d\Omega_{HI}$ calculated assuming isotropic angular correlation between positron and electron as presented in [23], except for $^{238}U + ^{181}Ta$ 635 keV.

[b]Cross section $d\sigma_{line}/d\Omega_{HI}$ calculated assuming back-to-back positron-electron angular correlation as presented in [23].

Figure 7:

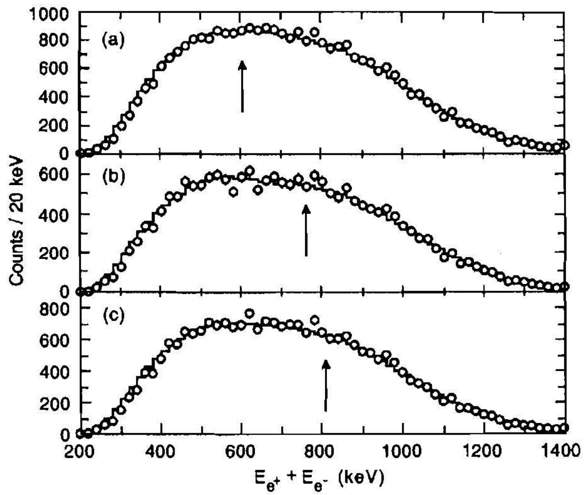

Figure 8: The positron-electron sum-energy spectra from $\mathrm{U}^{238} + \mathrm{Tr}^{232}$ collision obtained for wedge cuts associated with previously reported sum-energy lines of (a) 608 keV, (b) 760 keV, (c) 809 keV. Solid lines represent the spectra obtained from event mixing. Arrows indicate the positions of previously observed sum-energy lines.

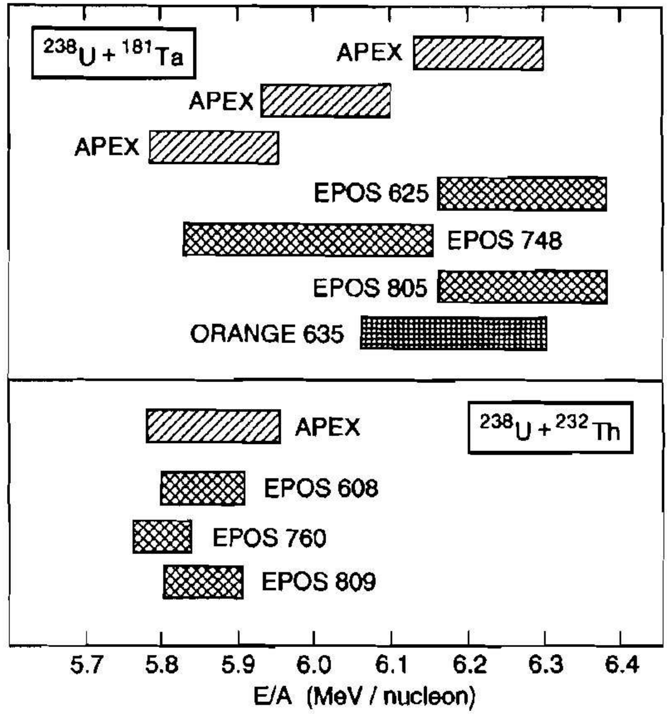

Figure 9: Energies covered by various experiments reporting e^+e^- coincidence lines in heavy ion collisions.

In summary, the present experiments have provided no evidence for the previously reported lines in positron-electron sum-energy spectra measured in $\mathrm{U}^{238} + \mathrm{Th}^{232}$ and $\mathrm{U}^{238} + \mathrm{Ta}^{181}$ systems. The upper limits for the line cross sections obtained from our data are, in all cases,significantly smaller than the value from the experiments reporting positive results, even when effects of a possible energy dependence of the cross section are considered. This new body of evidence must call into question the significance of earlier, positive, results.