An introduction to solar oscillations and helioseismology

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Abstract. Helioseismology offers unique possibilities for probing the detailed internal structure of a star and, in this way, constraining the physical properties of matter under stellar conditions. Here I provide a brief introduction to stellar structure and stellar oscillations, as well as to the techniques used in helioseismic analyses. In addition, I give a few examples of the results obtained from helioseismic investigations of solar structure.

1. INTRODUCTION

To learn about stellar interiors we need observables that reflect interior properties. Such information is provided, in a global sense, by the luminosity and surface temperature of a star, given that these quantities are determined by the stellar structure. Also, observations of the surface composition may provide some information about the internal properties, to the extent that the composition has been modified by mixing processes. However, the relations between stellar structure, let alone the physics of the stellar interior, and these observables are rather indirect and themselves involve complex and somewhat uncertain physical processes. In contrast, frequencies of stellar pulsation are related in a simple way to the properties of the stellar interior; furthermore, they can be determined with extremely high precision. With a sufficiently rich spectrum of oscillations, which is available in several very different types of stars, it is possible to obtain information about subtle aspects of conditions in localized parts of the star, and hence about the physical processes responsible for these conditions.

The extreme example of this type of diagnostics is the Sun where thousands of frequencies of acoustic oscillations are known with very high accuracy. The modes have periods between around 3 and 15 minutes and spatial scales ranging from spherically symmetric oscillations to modes with surface wavelengths of a few thousand kilometers. The amplitudes of individual modes are tiny: up to around $20\,\mathrm{cm\,s^{-1}}$ in velocity and a few parts per million in intensity. The variety of modes is such that the sound speed and other properties can be determined with high precision and resolution in much of the solar interior. Given the dependence of the sound speed on the thermodynamic state of the gas, particularly the adiabatic compressibility, this gives stringent constraints on the equation of state in the solar interior.

These aspects are discussed in detail in this volume. To set the scene the present paper provides an overview of solar modelling and of the observations and analysis that lead to the desired information about the thermodynamic properties of solar matter. Stellar modelling is discussed briefly in Section 2; for further details any of the large number of textbooks available, such as Kippenhahn & Weigert (1990), can be consulted. In Section 3 I present the basic properties of stellar acoustic oscillations, concentrating on how their frequencies are related to stellar properties; detailed descriptions of the theory of stellar oscillations was given by Unno *et al.* (1989) and Gough (1993). Observations of solar oscillations are considered briefly in Section 4, while Section 5 discusses a few of the results that have been obtained from helioseismic investigation of solar structure; there are several recent reviews on these issues (*e.g.* Christensen-Dalsgaard 2002). Finally, Section 6 discusses some of the problems and potentials of such investigations, including the extension to other stars with possibly more extreme physical conditions.

2. SOLAR MODELLING

It is obvious that stars, including the Sun, in principle are extremely complex physical systems, with a huge number of degrees of freedom. However, the description of stellar interiors are greatly simplified by the state of near equilibrium that is apparently satisfied at all levels, from the thermodynamic state to the overall energy budget of the star. The enormous disparity in scale between the size of the star and the mean free path of particles or photons in it ensures that local thermodynamical equilibrium is satisfied to a very high precision except in the stellar atmospheres, greatly simplifying the treatment of radiative transfer and nuclear reactions in stellar interiors and allowing the use of equilibrium thermodynamics. In general it enables a local description of the *microphysics*, *i.e.*, the properties of stellar matter and its interaction with radiation. On large scales, it seems that most stars are very nearly in hydrostatic and thermal equilibrium, with only slow changes with time as a result of the gradual depletion of nuclear fuel. These simplifications make it possible to construct simple, yet apparently fairly realistic, models of stellar interiors, an obvious prerequisite for the use of observations of stellar properties to study the underlying physics.

A remaining difficulty concerns possible hydrodynamical instabilities in stellar interiors, and other hydrodynamical effects such as might be induced, *e.g.*, by the rotation of stars. In most cases such effects are simply ignored, generally with little or no justification. An exception is convective instability, discussed below, which is certainly present in most stars with very significant effects on their structure and evolution. The commonly employed crude descriptions of the motion resulting from this instability is a serious uncertainty that must be taken into account in the interpretation and use of results of stellar modelling.

For future reference it is useful to discuss very briefly the relevant equations of stellar structure. I shall assume that the structure of the star is spherically symmetric and characterized by the distance r to the centre. This neglects the centrifugal force resulting from rotation, an excellent approximation for the slowly rotating Sun, and other effects that might lead to departures from spherical symmetry. Hydrostatic equilibrium requires a balance between the pressure gradient and gravity which may then be written as

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{Gm\rho}{r^2} \; , \tag{1}$$

where p is pressure, ρ is density, m is the mass of the sphere contained within r, and G is the gravitational constant. Also, obviously,

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho \; . \tag{2}$$

The gradient of temperature T is determined by the requirements of energy transport, from the central regions where nuclear reactions take place to the surface where the energy is radiated. The temperature gradient is conventionally written as

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \nabla \frac{T}{p} \frac{\mathrm{d}p}{\mathrm{d}r} \; , \tag{3}$$

where, obviously, $\nabla = \mathrm{d}\ln T / \mathrm{d}\ln p$. Its form depends on the mode of energy transport; for radiative transport

$$\nabla = \nabla_{\mathrm{rad}} \equiv \frac{3}{16\pi a\tilde{c}G} \frac{\kappa p}{T^4} \frac{L(r)}{m(r)} \; , \tag{4}$$

where κ is the opacity, L is the energy flow through the sphere of radius r, a is the radiation energy density constant and $\tilde{c}$ is the speed of light. The energy equation relates the energy generation to the energy flow and the change in the internal energy of the gas:

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \left[\rho\varepsilon - \rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e}{\rho}\right) + \frac{p}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} \right] \; ; \tag{5}$$

here ε is the rate of nuclear energy generation[1] per unit mass and unit time, e is the internal energy per unit volume and t is time.[2] Finally, we need to consider the rate of change of the composition, which controls stellar evolution. In a main-sequence star such as the Sun the dominant effect is the burning of hydrogen; however, we must also take into account the changes in composition resulting from diffusion and settling. The rate of change of the abundance X by mass of hydrogen is therefore given by

$$\frac{\partial X}{\partial t} = \mathscr{R}_{\mathrm{H}} + \frac{1}{r^2\rho}\frac{\partial}{\partial r}\left[r^2 \rho \left(D_{\mathrm{H}} \frac{\partial X}{\partial r} + V_{\mathrm{H}} X \right) \right] \; , \tag{6}$$

where $\mathscr{R}_{\mathrm{H}}$ is the rate of change resulting from nuclear reactions, D_{H} is the diffusion coefficient and V_{H} is the settling velocity.

To these basic equations we must add the description of the microphysics, most naturally, given the form of the equations, in terms of $(p, T, \{X_i\})$, where X_i are the

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[1] reduced for the emission of neutrinos which escape the star and hence do not contribute to the energy budget.

[2] For a star evolving in near thermal equilibrium the terms in the time derivatives are small.

abundances of the relevant elements. The determination of ρ, u and other required thermodynamical variables follows from the equation of state, extensively discussed elsewhere in this volume. The opacity (describing the absorption and scattering of radiation by the constituents of the gas) is most naturally obtained as $\kappa = \kappa(\rho, T, \{X_i\})$. Evidently the calculation of the opacity requires knowledge of the relevant atomic parameters; however, the opacity also depends crucially on the thermodynamic state of the gas, through the occupation of atomic ionization and excitation levels and the perturbation of atomic states by neighbouring particles. Similarly, the rates of nuclear reactions depend not only on the nuclear parameters but also, as discussed by Shaviv in this volume, on the interactions between the particles in the gas which leads to partial screening of the repulsive Coulomb potential between the nuclei. Also, calculation of the diffusion and settling coefficients depends on the thermodynamic state.

I have so far ignored the convective instability. This sets in if the density decreases more slowly with position than for an adiabatic change, *i.e.*,

$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}\ln p} < \frac{1}{\gamma_1} , \tag{7}$$

where $\gamma_1 = (\partial \ln p / \partial \ln \rho)_{\mathrm{ad}}$, the derivative being taken for an adiabatic change. In stellar modelling this condition is often replaced by

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln p} \equiv \nabla > \nabla_{\mathrm{ad}} \equiv \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln p}\right)_{\mathrm{ad}} \tag{8}$$

which is equivalent in the case of a uniform composition.[3] Thus a layer is convectively unstable if the radiative gradient ∇_{rad} (cf. Eq. 4) exceeds ∇_{ad}. In this case convective motion sets in, with hotter gas rising and cooler gas sinking, both contributing to the energy transport towards the surface. The condition that the combined radiative and convective energy transport through a surface of radius r match the luminosity then in principle defines a condition for the average temperature gradient which can be written as

$$\nabla = \nabla_{\mathrm{conv}}(\rho, T, L, \ldots) . \tag{9}$$

In practice this relation depends on the details of the convective flow which is very likely turbulent and represents conditions over a range of positions in the star; also, motion is inevitably induced outside the immediate unstable region.

Substantial progress has been made towards the modelling of time-dependent convection in the near-surface regions (*e.g.* Stein & Nordlund 1989, 1998; Robinson *et al.* 2003). However, for the general treatment of convection in stellar modelling a simpler prescription is needed. In the so-called mixing-length models this is typically based on rough models of the convective eddies, characterized by a length scale ℓ usually taken as a multiple $\alpha_{\mathrm{ML}} H_p$ of the pressure scale height H_p (*e.g.* Böhm-Vitense 1958). This can be generalized to include non-local effects (*e.g.* Spiegel 1963; Gough 1977; Balmforth

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[3] For the complications arising when composition is not uniform, see for example Kippenhahn & Weigert (1990).

1992a); treatments that take some aspects of the spectrum of turbulence into account have also been developed (*e.g.* Canuto & Mazzitelli 1991). A general feature of these treatments of convection, confirmed by the hydrodynamical simulations, is that except in a thin region near the surface the resulting temperature gradient exceeds the adiabatic value by a very small amount,[4]

$$\nabla_{\rm conv} \simeq \nabla_{\rm ad} = \nabla_{\rm ad}(p, T, \{X_i\}) \; , \tag{10}$$

emphasizing that $\nabla_{\rm ad}$ is a thermodynamical quantity.

The Sun is unique amongst stars in that we have fairly accurate measurements of its mass, radius and energy output, as well as a precise age inferred from the radioactively determined ages of meteorites. Furthermore spectral analysis has provided determinations of the solar surface abundances, although, as indicated in Section 5, spectroscopic analysis of the solar atmospheric abundance is complicated by uncertainties in the modelling of the solar atmosphere; additional information about abundances of refractory elements comes from the analysis of the composition of meteorites (*e.g.* Anders & Grevesse 1989). An important exception is helium: despite the fact that helium was first detected in spectral lines arising in the higher parts of the solar atmosphere, conditions in this region are so uncertain that no precise measurement of the helium abundance can be made from spectroscopy. Since hydrogen and helium are by far the most abundant elements the composition of stellar matter is typically characterized by the abundances by mass of hydrogen, denoted X, helium (Y) and elements heavier than helium (Z) with, obviously, $X + Y + Z = 1$. However, as discussed later the relative composition of these 'heavy' elements is also important, particularly for the determination of the opacity.

Models of the Sun should obviously be consistent with the measurements of surface radius, luminosity and composition. In practice this is achieved by evolving a model of solar mass (assuming that massloss since the formation of the Sun has been negligible; see, however, Sackmann & Boothroyd 2003) to the present age of the Sun. To obtain the right radius, luminosity and present surface composition three parameters are adjusted. One of them, *e.g.* the constant $\alpha_{\rm ML}$ in the mixing-length description of convection, determines the efficacy of convection and hence the superadiabatic gradient $\nabla - \nabla_{\rm ad}$ required for energy transport in the thin significantly superadiabatic gradient of the convection zone. The initial helium abundance Y_0, which is not constrained from spectroscopic measurements, is adjusted to match the observed surface luminosity, and the initial heavy-element abundance Z_0 is chosen to obtain the observed ratio $(Z/X)_{\rm s}$ between the surface heavy-element and hydrogen abundances.

The calibration of the convective treatment serves to determine the adiabat of the convection zone which in turn ensures that the model has the correct radius. It follows that, after calibration, models computed with different treatments of convection yield essentially the same structure except in the thin region of significant superadiabaticity. This is illustrated in Fig. 1. The fact that the adiabat of the bulk of the solar convection

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[4] Crudely speaking this follows from the fact that the internal energy density is so high, except near the surface, that only a very small excess in the energy density and hence in temperature above the adiabatic value suffices for the required energy transport.

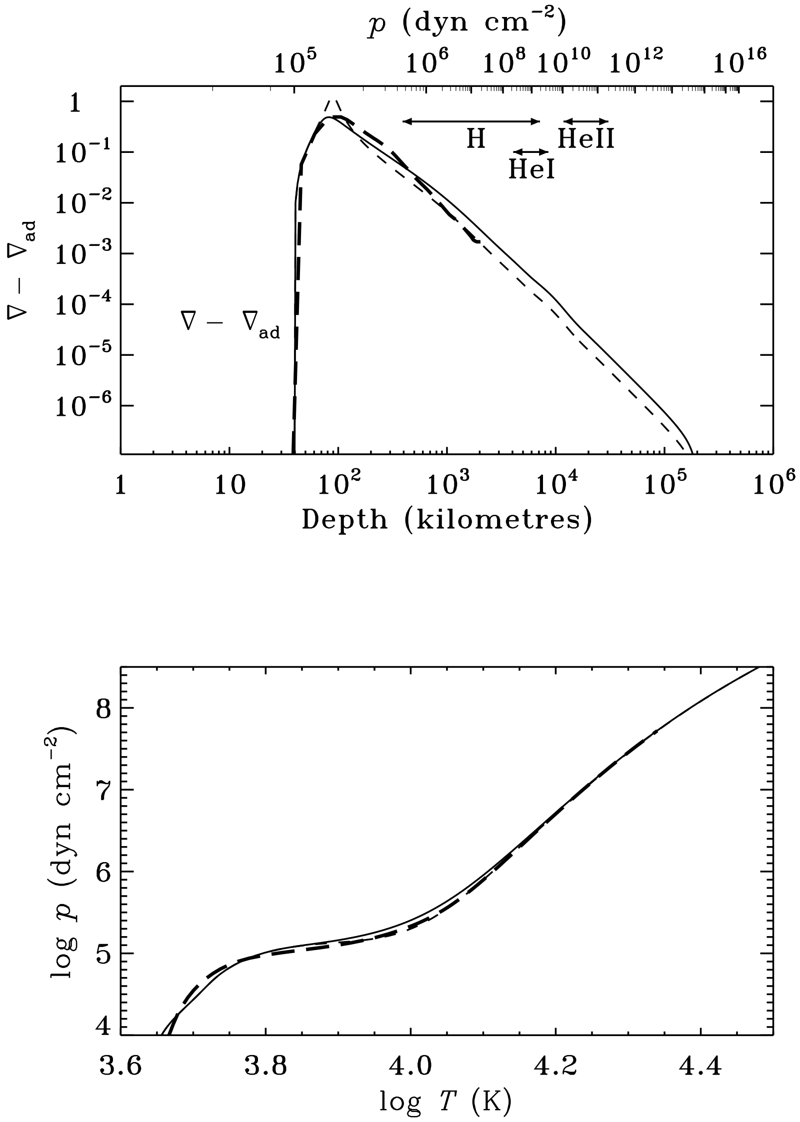

FIGURE 1. Properties of the solar convection zone. In the top panel the lower abscissa is depth below the point in the atmosphere where the temperature equals the effective temperature, whereas the upper abscissa is pressure p. The solid curve shows the superadiabatic temperature gradient $\nabla - \nabla_{ad}$, and the horizontal arrows indicate the locations of the ionization zone of hydrogen and the first and second ionization zones of helium, extending between the points of 10 and 90 % ionization, for Model S of Christensen-Dalsgaard *et al.* (1996), which used the Böhm-Vitense (1958) mixing-length theory. In addition, the short-dashed curve shows $\nabla - \nabla_{ad}$ for a model using the Canuto & Mazzitelli (1991) treatment of convection, also calibrated to yield a model of solar surface radius, and the heavy long-dashed curve shows $\nabla - \nabla_{ad}$ in the average model resulting from hydrodynamical simulations (Stein & Nordlund 1989, 1998). (Adapted from Gough & Weiss 1976.) The lower panel shows the relation between temperature and pressure in the upper parts of the same three models, using the same line styles.

zone is thus defined is evidently of substantial importance in the tests of equations of state from studies of solar properties.

3. PROPERTIES OF OSCILLATIONS

3.1. Basic equations of adiabatic oscillations

In principle stellar oscillations constitute a complex hydrodynamical problem, involving issues of mode excitation and damping as well as the determination of the oscillation frequencies. However, for the present purpose of discussing the diagnostic potential of the observed solar oscillations, where we are principally interested in the determination of the oscillation frequencies, very substantial simplifications are possible. The amplitudes of the observed modes are so small that the oscillations can be treated as small perturbations around an equilibrium state, corresponding to the model discussed in the preceding section. Thus, for example, the pressure at some position $\mathbf{r}$ and time t is written as $p(\mathbf{r},t) = p_0(r) + p'(\mathbf{r},t)$, where, as indicated, the equilibrium pressure p_0 is assumed to be spherically symmetric, *i.e.*, depending only on r. The flow is described by the velocity $\mathbf{v}$ or the displacement $\boldsymbol{\delta}\mathbf{r}$, related by

$$\mathbf{v} = \frac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t} . \tag{11}$$

In addition to the local, or *Eulerian*, perturbations p' etc., it is often convenient to consider also *Lagrangian* perturbations following the motion, *e.g.*, δp. It is straightforward to demonstrate that

$$\delta p = p' + \boldsymbol{\delta}\mathbf{r} \cdot \nabla p_0 . \tag{12}$$

The equations for the perturbations are obtained by inserting these expansions in the general hydrodynamical equations, using that the equilibrium quantities satisfy the equations of stellar structure and retaining only terms linear in the perturbations. For the *equation of continuity* the result is

$$\frac{\partial \rho'}{\partial t} + \operatorname{div}(\rho_0 \mathbf{v}) = 0 , \tag{13}$$

or, by using Eq. (11) and integrating with respect to time

$$\rho' + \operatorname{div}(\rho_0 \boldsymbol{\delta}\mathbf{r}) = 0 . \tag{14}$$

The *equations of motion* become

$$\rho_0 \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 , \tag{15}$$

where $\mathbf{g}_0$ and $\mathbf{g}'$ are the equilibrium and perturbed gravitational accelerations, respectively, the latter conveniently expressed as $\mathbf{g}' = -\nabla\Phi'$, where the perturbed gravitational potential satisfies Poisson's equations,

$$\nabla^2 \Phi' = 4\pi G \rho' . \tag{16}$$

To close the system we must relate p' and ρ'; this in general requires consideration of the energy equation and hence the full complexities of the perturbation in the energy transport and generation. However, in most of the star the thermal timescale far exceeds the typical periods of oscillation. Thus it is a good approximation to assume that the oscillations take place adiabatically without energy exchange, resulting in a thermodynamical relation between the pressure and density perturbations. This is most conveniently written in terms of the Lagrangian perturbations, as

$$\delta p = \frac{\gamma_{1,0} p_0}{\rho_0} \delta \rho \ . \tag{17}$$

Solar rotation is so slow that the equilibrium structure can be assumed to be spherically symmetric to a very good approximation. Then the perturbation equations are separable in spherical polar coordinates (r, θ, ϕ), where θ is colatitude (*i.e.*, the angle from the polar axis), and ϕ is longitude. Thus the dependence of (θ, ϕ) is written as a function $f(\theta, \phi)$ which, to obtain separated equations, must satisfy

$$\nabla_{\rm h}^2 f = -\frac{1}{r^2} \Lambda f \ , \tag{18}$$

for some eigenvalue Λ, where $\nabla_{\rm h}^2$ is the (θ, ϕ)-part of the Laplace operator. A solution to this equation is

$$f(\theta, \phi) = (-1)^m c_{lm} P_l^m(\cos\theta) \exp(im\phi) \equiv Y_l^m(\theta, \phi) \ , \tag{19}$$

where P_l^m is a Legendre function and Y_l^m is a spherical harmonic; here c_{lm} is a normalization constant such that the integral of $|Y_l^m|^2$ over the unit sphere is 1. Also,

$$\Lambda = l(l+1) \ , \tag{20}$$

where l is a non-negative integer and

$$|m| \leq l \ . \tag{21}$$

Y_l^m is characterized by its *degree* l and its *azimuthal order* m; it follows from Eqs (18) and (20) that l is related to the local horizontal wave number $k_{\rm h}$ by

$$k_{\rm h}^2 = \frac{l(l+1)}{r^2} \ . \tag{22}$$

Since the equilibrium structure is, to a very good approximation, independent of time on the timescale of the oscillations we can also separate the time dependence as $\exp(-i\omega t)$. Note that these expansions are written on complex form; the physical variables are obtained by taking the real parts of the resoluting solutions. It follows that the dependence on ϕ and time can be written, apart from a phase, as $\cos(m\phi - \omega t)$.

The separation of a scalar variable may be written as, *e.g.*,

$$p'(r, \theta, \phi, t) = \sqrt{4\pi}\, \tilde{p}'(r) Y_l^m(\theta, \phi) \exp(-i\omega t) \ , \tag{23}$$

From the momentum equation it may be shown that the (physical) displacement is given by

$$\begin{aligned} \boldsymbol{\delta}\mathbf{r} \;=\; & \sqrt{4\pi}\Re\left\{\left[\tilde{\xi}_r(r)Y_l^m(\theta,\phi)\mathbf{a}_r\right.\right. \\ & \left.\left.+\tilde{\xi}_{\rm h}(r)\left(\frac{\partial Y_l^m}{\partial\theta}\mathbf{a}_\theta + \frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\mathbf{a}_\phi\right)\right]\exp(-i\omega t)\right\} \; , \end{aligned} \tag{24}$$

where $\Re$ indicates the real part, $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ are unit vectors in the r, θ and ϕ directions and

$$\tilde{\xi}_{\rm h}(r) = \frac{1}{r\omega^2}\left(\frac{1}{\rho_0}\tilde{p}' + \tilde{\Phi}'\right) \; . \tag{25}$$

Thus the problem of determining the oscillations of the model is reduced to solving a set of ordinary differential equations for the amplitude functions $\tilde{p}'(r)$, $\tilde{\xi}_r(r)$, etc. Using that, according to the adiabatic approximation (Eq. 17),

$$\rho' = \frac{\rho}{\gamma_1 p}p' + \rho\xi_r\left(\frac{1}{\gamma_1 p}\frac{{\rm d}p}{{\rm d}r} - \frac{1}{\rho}\frac{{\rm d}\rho}{{\rm d}r}\right) \; , \tag{26}$$

where I dropped the ‘0’ on equilibrium quantities and the tildes, we obtain

$$\frac{{\rm d}\xi_r}{{\rm d}r} = -\left(\frac{2}{r} + \frac{1}{\gamma_1 p}\frac{{\rm d}p}{{\rm d}r}\right)\xi_r + \frac{1}{\rho c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)p' + \frac{l(l+1)}{\omega^2 r^2}\Phi' \; , \tag{27}$$

$$\frac{{\rm d}p'}{{\rm d}r} = \rho(\omega^2 - N^2)\xi_r + \frac{1}{\gamma_1 p}\frac{{\rm d}p}{{\rm d}r}p' - \rho\frac{{\rm d}\Phi'}{{\rm d}r} \; , \tag{28}$$

and

$$\frac{1}{r^2}\frac{\rm d}{{\rm d}r}\left(r^2\frac{{\rm d}\Phi'}{{\rm d}r}\right) = 4\pi G\left(\frac{p'}{c^2} + \frac{\rho\xi_r}{g}N^2\right) + \frac{l(l+1)}{r^2}\Phi' \; . \tag{29}$$

Here I introduced the characteristic acoustic frequency S_l by

$$S_l^2 = \frac{l(l+1)c^2}{r^2} = k_{\rm h}^2 c^2 \; . \tag{30}$$

where $c^2 = \gamma_1 p/\rho$ is the square of the adiabatic sound speed, and the buoyancy frequency N, given by

$$N^2 = g\left(\frac{1}{\gamma_1 p}\frac{{\rm d}p}{{\rm d}r} - \frac{1}{\rho}\frac{{\rm d}\rho}{{\rm d}r}\right) \; . \tag{31}$$

Equations (27), (28) and (29) constitute a fourth-order system for the amplitude functions $\{\xi_r, p', \Phi', {\rm d}\Phi'/{\rm d}r\}$.

At the centre, $r = 0$, the solutions must satisfy two regularity conditions. At the surface Φ' and its derivative must match continuously to the decreasing solution in the vacuum outside the star, leading to

$$\frac{{\rm d}\Phi'}{{\rm d}r} + \frac{l+1}{r}\Phi' = 0 \qquad \text{at } r = R \; . \tag{32}$$

A second condition is formally obtained by assuming that there is no pressure perturbation at the perturbed surface of the star, *i.e.*,

$$\delta p = p' + \xi_r \frac{\mathrm{d}p}{\mathrm{d}r} = 0 \qquad \text{at } r = R \,, \tag{33}$$

In practice, the 'surface' must be defined as the outermost point of the model, which is obviously somewhat arbitrary. Thus more sophisticated conditions are used in calculations of stellar oscillations. These show that trapped modes are restricted to have frequencies below the acoustical cut-off frequency $\omega_{\rm ac}$ (*cf.* Eq. 40); at frequencies well below this value Eq. (33) is still approximately satisfied. In any case, as discussed below, the properties of the oscillations in the superficial layers remain a serious problem in the computation of solar oscillation frequencies.

The system of differential equations together with the boundary conditions has non-trivial solutions only for discrete eigenvalues. The equations and boundary conditions are independent of the azimuthal order m and so, therefore are the eigenfrequencies. This follows obviously from the assumed spherical symmetry of the equilibrium structure: the definition of m depends on the the orientation of the axis of the spherical harmonics which can have no physical effect for a spherically symmetric star.[5] Computed frequencies for a solar model, as functions of the degree l are illustrated in Fig. 2. The observed frequencies in the Sun, greater than around $1000\,\mu$Hz, clearly correspond to acoustic modes of fairly high order or high degree or, at high degree, to surface gravity (f) modes.

From Eqs (33), (1) and (25), neglecting Φ',[6] it follows that

$$\frac{\xi_{\rm h}(R)}{\xi_r(R)} = \frac{g_{\rm s}}{R\omega^2} \,, \tag{34}$$

where $g_{\rm s}$ is the surface gravity. For modes of high frequency, typical of those observed in the Sun, and low or moderate degree, this shows that the motion in the atmosphere is predominantly in the radial direction.

3.2. Excitation of solar oscillations

The description of solar oscillations made so far evidently provides no information about the origin of the observed solar oscillations: given the adiabatic approximation and suitable outer boundary conditions the modes preserve energy and hence neither grow nor decay in amplitude. A more complete treatment would include the energy equation and would then determine whether the modes are stable or unstable; formally,

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[5] Rotation introduces a frequency splitting according to m which has been used to obtain detailed information about solar internal rotation and its variation with time. For a recent review, see Thompson *et al.* (2003).

[6] This so-called *Cowling approximation* (Cowling 1941) is quite reasonable for high-order or high-degree modes, such as observed in the Sun.

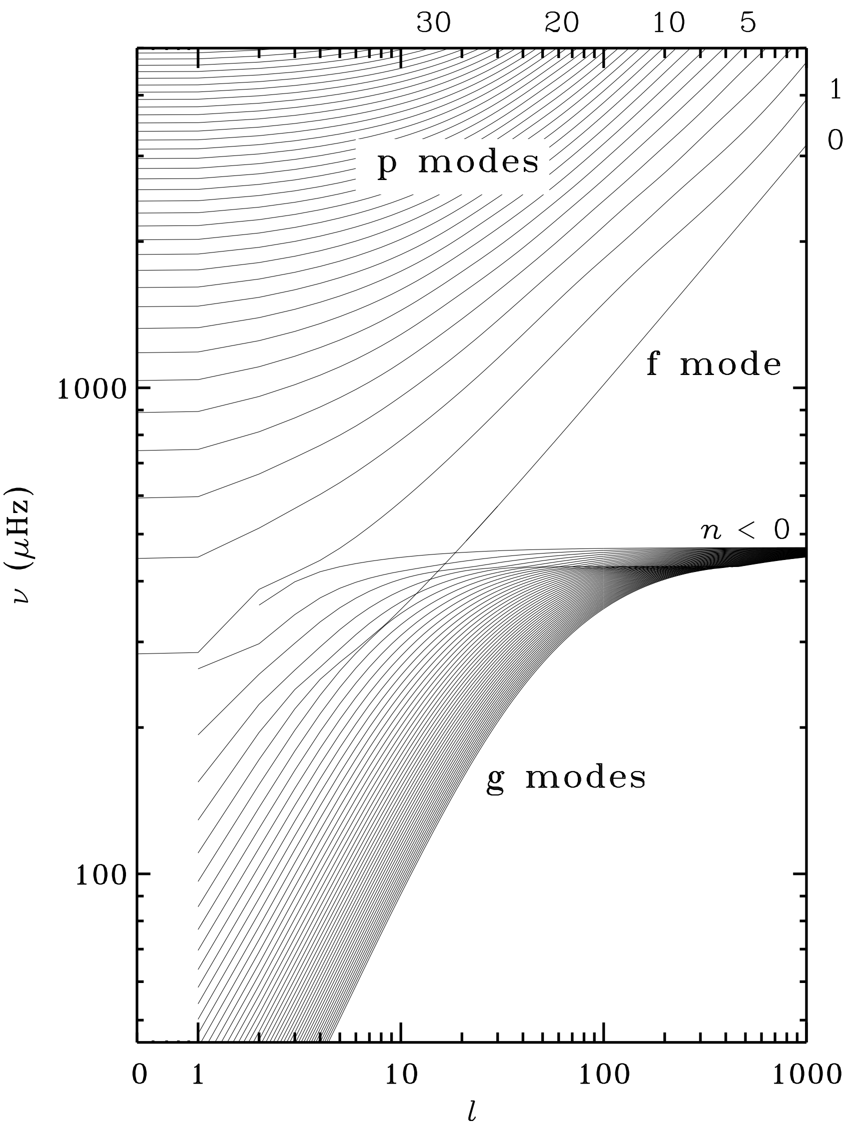

FIGURE 2. Computed frequencies for a model of the present Sun, as functions of degree l. For clarity, the points corresponding individual degrees have been connected; although the abscissa is generally logarithmic, the left-hand edge corresponds to radial modes, with $l = 0$. The radial order n is indicated for some of the modes; conventionally $n > 0$ is used for acoustic (or p) modes, $n = 0$ for the so-called f modes which at moderate and high degree have the character of surface gravity waves, and $n < 0$ for internal gravity (or g) modes. The order is defined such that, in most cases, $|n|$ gives the number of nodes in the radial direction.

the eigenfrequencies ω would then be complex, with positive or negative imaginary parts corresponding to growing or decaying modes, respectively. The treatment needs to take into account the effects of convection on the perturbation in the flux; an additional important fact is the perturbation in the turbulent pressure, whose phase may be such as to damp or drive the oscillation. The computation of such time-dependent effects of convection is highly uncertain; however, using a time-dependent version of a non-local mixing-length description Balmforth (1992a) concluded that the modes observed in the Sun are likely damped.

If the modes are indeed damped, they must be driven by external forcing. A likely source is the near-surface convection which reaches close to sonic speed and hence is an efficient emitter of acoustic noise (Lighthill 1952; Stein 1967). This essentially stochastic source of sound waves couples to the normal modes of the Sun and excites them to the observed amplitudes. This stochastic excitation is consistent with the observed statistical properties of the amplitude distribution (*e.g.* Chaplin *et al.* 1997; Chang & Gough 1998).

An early analysis of the stochastic excitation of solar modes was carried out by Goldreich & Keeley (1977). A more detailed analysis, although still based on a mixing-length description of convection, was provided by Balmforth (1992b). It was pointed out by Brown (1991) that since the acoustic emission depends on the turbulent velocity to a high power the excitation is probably fairly localized and confined to a small volume near the top of the convection zone. Observational evidence for such localized excitation was obtained by, for example, Goode, Gough & Kosovichev (1992) and Rimmele *et al.* (1995) who found an association between the acoustic exciting events and the dark intergranular lanes associated with downdrafts. A detailed model of the excitation caused by such downdrafts was developed by Rast (1999). Also, hydrodynamical simulations have demonstrated that convection provides the required energy input to the modes, determined observationally from the amplitude and width of the peaks on the observed power spectra[7] (*e.g.* Stein & Nordlund 2001; Stein *et al.* 2004). Since this driving mechanism operates on a small spatial scale and over a range of timescales, it is expected to yield a spectrum of oscillations covering a broad range of degrees and frequencies, as observed.

The properties of the excitation are reflected in the detailed shapes of the observed peaks in the oscillation power spectra, which show distinct asymmetries (Duvall *et al.* 1993), related to the location of the source (see also, for example, Gabriel 1993; Abrams & Kumar 1996; Roxburgh & Vorontsov 1997; Nigam & Kosovichev 1998; Rast & Bogdan 1998; Rosenthal 1998). Analyses of this asymmetry have confirmed that the source of the excitation is indeed confined to the upper part of the convection zone (*e.g.* Kumar & Basu 1999, 2000; Nigam & Kosovichev 1999; Chaplin & Appourchaux 1999). Additional information about the properties of the acoustic sources has been obtained from observations of resonances, or ‘pseudo-modes’ at frequencies above the acoustical cut-off frequency (*cf.* Eq. 40) (*e.g.* Kumar *et al.* 1990; Kumar & Lu 1991; Kumar 1994; Kumar *et al.* 1994).

The stochastic excitation by convection is expected to operate in any star with vigorous near-surface convection. Very accurate observations over the past few years have in fact found oscillations in a number of stars with outer convection zones, including also subgiants and giants (for a recent brief review, see Bedding & Kjeldsen 2003).

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[7] The spectral width of a damped and stochastically excited oscillator is determined by the damping rate of the undriven oscillator (*e.g.* Batchelor 1956; Christensen-Dalsgaard, Gough & Libbrecht 1989).

3.3. The dependence of frequencies on solar properties

From the point of view of investigating solar properties based on the oscillation frequencies the dependence of the frequencies on solar structure is obviously crucial. This is implicitly given by the coefficients in Eqs (27) – (29). However, these are constrained by the fact that the equilibrium model satisfies the equations of stellar structure. In fact, given the density distribution $\rho(r)$ the mass distribution $m(r)$ can be obtained from Eq. (2); the pressure distribution $p(r)$ then follows by integration of Eq. (1). If in addition $\gamma_1(r)$ is given, it is easy to show that all required coefficients can be determined. Thus, from the point of view of computing adiabatic oscillation frequencies, the model is completely specified by specifying $\rho(r)$ and $\gamma_1(r)$. It follows that analysis of the oscillation frequencies can at most give information about these 'dynamical' variables. In particular, no direct information is obtained about temperature; of course, if an equation of state is assumed and if the composition is given, the temperature can be inferred from p, ρ and the composition. Other pairs of functions equivalent to (ρ, γ_1) can be substituted; since the observed modes are mostly acoustic modes their properties are to a large extent determined by the adiabatic sound speed, suggesting the pair (c^2, ρ) which is often used.

Analysis of the observed frequencies show that present solar models are likely to be relatively close to the true solar structure. In this case changes in the computed frequencies can be obtained by linearizing the oscillation equations around a reference solar model. The analysis is greatly simplified by noting that the adiabatic oscillation frequencies satisfy a variational principle (Chandrasekhar 1964). We let $\delta\omega_{nl} = \omega_{nl}^{(\mathrm{obs})} - \omega_{nl}^{(\mathrm{mod})}$ be the difference between the observed frequencies and the frequencies of the reference model; also, we characterize solar structure by the pair (c^2, ρ) and let $\delta_r c^2 = c^2 - c^2_{\mathrm{mod}}$ and $\delta_r \rho = \rho - \rho_{\mathrm{mod}}$ be the differences between the solar and model values of c^2 and ρ, at fixed r. Then we can express the frequency differences as

$$\frac{\delta\omega_{nl}}{\omega_{nl}} = \int_0^R \left[K^{nl}_{c^2,\rho}(r) \frac{\delta_r c^2}{c^2}(r) + K^{nl}_{\rho,c^2}(r) \frac{\delta_r \rho}{\rho}(r) \right] \mathrm{d}r + Q_{nl}^{-1} \mathscr{G}(\omega_{nl}) , \qquad (35)$$

(*e.g.* Dziembowski, Pamyatnykh & Sienkiewicz 1990; Gough & Thompson 1991), where the *kernels* $K^{nl}_{c^2,\rho}$ and K^{nl}_{ρ,c^2} are determined by the eigenfunctions computed for the reference model.

In Eq. (35) the last term describes the effects of near-surface errors in the frequency calculation. The expression in terms of the kernels assumes that the frequencies of solar oscillation are correctly described by the assumed adiabatic pulsation equations. In fact, in the near-surface layers, where the thermal timescale is comparable to the pulsation period, the adiabatic approximation certainly breaks down. Also, the frequencies may be affected by other uncertain aspects of the physics of the model and the oscillations, not least the treatment of convection and its interaction with the pulsations. It may be shown that these effects, at least for modes of low or moderate degree, can be described by the form indicated, where $\mathscr{G}(\omega)$ is a function of frequency alone that depends on the unknown physical processes. Q_{nl} is a simple scaling that depends on the inertia of the modes, normalized to be unity for radial modes; it compensates for the fact, discussed below, that higher-degree acoustic modes occupy a smaller fraction of the solar interior and hence are more strongly affected by near-surface errors (*e.g.* Christensen-Dalsgaard & Berthomieu 1991).

It is evident that γ_1 is determined by the equation of state as $\gamma_1 = \gamma_1(p, \rho, Y, Z)$ where, as before, we characterize the composition by the abundances X, Y and Z of hydrogen, helium and heavier elements. Furthermore, using that $c^2 = \gamma_1 u$, where $u = p/\rho$, and utilizing the equations of stellar structure, Eq. (35) can be written as

$$\frac{\delta\omega_{nl}}{\omega_{nl}} = \int_0^R K_{u,Y}^{nl}(r)\frac{\delta_r u}{u}(r)\mathrm{d}r + \int_0^R K_{Y,u}^{nl}(r)\delta_r Y(r)\mathrm{d}r + \int_0^R K_{c^2,\rho}^{nl}(r)\left(\frac{\delta\gamma_1}{\gamma_1}\right)_{\mathrm{int}}\mathrm{d}r + Q_{nl}^{-1}\mathscr{G}(\omega_{nl}) \tag{36}$$

(Basu & Christensen-Dalsgaard 1997), where for simplicity I neglected the effect of the difference in Z; here $(\delta\gamma_1/\gamma_1)_{\mathrm{int}}$ is the *intrinsic* difference between the solar and the model equation of state, *i.e.*, the difference at fixed p, ρ and composition. Evidently, constraints on $(\delta\gamma_1/\gamma_1)_{\mathrm{int}}$ would be an important contribution to the investigations of the equation of state through helioseismology.

A great deal of insight into the properties of the solar modes can be obtained from a very simple asymptotic analysis. The modes are predominantly standing sound waves which locally satisfy the dispersion relation

$$\omega^2 = c^2|\mathbf{k}|^2 = c^2\left[k_r^2 + \frac{l(l+1)}{r^2}\right] ; \tag{37}$$

here $\mathbf{k}$ is the local wave vector which I have separated into a radial component $k_r\mathbf{a}_r$ and a horizontal component, the length k_h of which satisfies Eq. (22). It follows that k_r is given by

$$k_r = \left[\frac{\omega^2}{c^2} - \frac{l(l+1)}{r^2}\right]^{1/2} , \tag{38}$$

for $r \geq r_\mathrm{t}$, where the *inner turning point* radius r_t is determined by

$$\frac{c(r_\mathrm{t})}{r_\mathrm{t}} = \frac{\omega}{\sqrt{l(l+1)}} ; \tag{39}$$

at $r = r_\mathrm{t}$ $k_r = 0$ and the waves travel horizontally, corresponding to a total internal reflection at this point, and below it the wave is evanescent, decaying exponentially.

This simple description does not provide reflection of the waves near the solar surface. This takes place where the wavelength becomes comparable with the density scale height H and where therefore the approximation of plane sound waves in a locally uniform medium no longer holds. It may be shown, from a simple analysis of waves in an isothermal atmosphere or from a more complete asymptotic analysis (*e.g.* Deubner & Gough 1984), that reflection takes place only when the frequency ω is below the *acoustical cut-off frequency* ω_{ac}; in the case of an isothermal atmosphere it is given by

$$\omega_{\mathrm{ac}} = \frac{c}{2H} . \tag{40}$$

In the solar case this corresponds to a cyclic frequency of around 5.3 mHz. Waves with higher frequencies can propagate out through the solar atmosphere, hence loosing energy and being strongly damped. Thus the acoustical cut-off frequency provides an upper limit to the frequencies of standing waves in a star.

The condition for a standing wave must be that the phase change in the radial direction between $r_{\rm t}$ and the near-surface reflection is an integer times π, with a possible correction for the phase changes at the surface and the inner turning point; thus

$$\int_{r_{\rm t}}^{R} k_r {\rm d}r = (n+\alpha)\pi \; , \tag{41}$$

where α accounts for the phase change at the surface (and at the inner turning point). Using Eq. (38) it follows that the frequencies satisfy

$$\int_{r_{\rm t}}^{R} \left(1 - \frac{L^2 c^2}{\omega^2 r^2}\right)^{1/2} \frac{{\rm d}r}{c} = \frac{[n+\alpha(\omega)]\pi}{\omega} \; , \tag{42}$$

where $L = \sqrt{l(l+1)}$ and I have made explicit that the surface phase change in general depends on frequency. A relation of this form, known as the *Duvall law*, was first obtained for the solar oscillation frequencies on the basis of observational data by Duvall (1982). Although derived here on the basis of highly simplified arguments, it can be obtained from a more rigorous asymptotic analysis of the full oscillation equations (*e.g.*, Gough 1993).

As discussed in connection with Eq. (35) it is useful to analyse the observed frequencies in terms of differences relative to a reference model. Thus we can linearize Eq. (42) in terms of a correction $\delta_r c$ to the sound speed and a correction $\delta\alpha$ to the phase function. The result can be written

$$\mathcal{S}_{nl} \frac{\delta\omega_{nl}}{\omega_{nl}} \simeq \mathcal{H}_1\left(\frac{\omega_{nl}}{L}\right) + \mathcal{H}_2(\omega_{nl}) \; , \tag{43}$$

where

$$\mathcal{S}_{nl} = \int_{r_{\rm t}}^{R} \left(1 - \frac{L^2 c^2}{r^2 \omega_{nl}^2}\right)^{-1/2} \frac{{\rm d}r}{c} - \pi \frac{{\rm d}\alpha}{{\rm d}\omega} \; , \tag{44}$$

$$\mathcal{H}_1(w) = \int_{r_{\rm t}}^{R} \left(1 - \frac{c^2}{r^2 w^2}\right)^{-1/2} \frac{\delta_r c}{c} \frac{{\rm d}r}{c} \; , \tag{45}$$

and

$$\mathcal{H}_2(\omega) = \frac{\pi}{\omega} \delta\alpha(\omega) \tag{46}$$

(Christensen-Dalsgaard, Gough & Pérez Hernández 1988). These expressions are clearly quite similar to Eq. (35) although, in this simple asymptotic description, there is no dependence on density. In particular, the effect of the phase difference, in $\mathcal{H}_2(\omega)$ corresponds to the term $\mathcal{G}(\omega)$ included in the non-asymptotic expression. Observationally, an expression of the functional form in Eq. (43) can be fitted to differences between observed and model frequencies, to determine the functions $\mathscr{H}_1$ and $\mathscr{H}_2$ (Christensen-Dalsgaard, Gough & Thompson 1989).

A given mode is essentially only sensitive to solar conditions outside its inner turning point, *i.e.* for $r \geq r_t$. For the observed range of degrees, from 0 to more than 1000, and frequencies in the vicinity of 3 mHz r_t ranges from the solar centre to very near the solar surface. It is to a large extent this variation in penetration depth and sensitivity which allows the observed frequencies to be combined in such a way as to provide localized information about the solar interior. An explicit illustration of this follows from the Duvall law or its differential form, Eqs (42) or (45). These are essentially in the form of Abel integral equations which may be solved for c, or $\delta_r c/c$, if the right-hand side of Eq. (42) or $\mathscr{H}_1$ have been determined from observations (*e.g.* Gough 1984; Christensen-Dalsgaard *et al.* 1985; Christensen-Dalsgaard *et al.* 1989; Vorontsov & Shibahashi 1991; Marchenkov, Roxburgh & Vorontsov 2000; Vorontsov, this volume).

The information content in the oscillation frequencies goes beyond such simple asymptotic descriptions, however. As discussed by Houdek (this volume) an interesting example is the effect of relatively sharp features, relative to the wavelength of the modes, in the equilibrium structure, which give rise to a signal in the frequencies determined by the phase of the modes at the features. The localized decrease in γ_1 associated with the second ionization of helium is one such feature, and the resulting signal may be used to determine the helium abundance in the convective envelope (*e.g.* Vorontsov *et al.* 1991; Pérez Hernández & Christensen-Dalsgaard 1994; Basu & Antia 1995).

3.4. Helioseismic inversion

Although asymptotic techniques are powerful, intuitive and efficient, they obviously do not make the fullest possible use of the information contained in the oscillation frequencies. Similarly, the inferences based on sharp features discussed by Houdek (this volume), while very powerful, are also limited to specific aspects of the solar interior. More general inferences that do not depend on the asymptotic properties of the modes can be made through inverse analyses of linearized relations such as Eq. (35). This is discussed in detail by Gough & Thompson (1991) and Gough (1996). Here I summarize some important features. Further detail on inversion procedures and results is provided by Vorontsov (this volume).

The goal of the analysis is to obtain localized information about the corrections to the solar model, taking into account also the errors in the data. To be specific, I consider inversion to determine $\delta_r c^2/c^2$. Most inversion techniques effectively correspond to making linear combinations of the differences $\delta\omega_{nl}/\omega_{nl}$; the result, attempting to infer the solution at some point $r = r_0$, can be written

$$\begin{aligned} \left\langle \frac{\delta_r c^2}{c^2} \right\rangle (r_0) &= \sum_i c_i(r_0) \frac{\delta\omega_i}{\omega_i} \\ &= \int_0^R \mathscr{K}_{c^2,\rho}(r_0,r) \frac{\delta_r c^2}{c^2}(r)\mathrm{d}r + \int_0^R \mathscr{C}_{\rho,c^2}(r_0,r) \frac{\delta_r \rho}{\rho}(r)\mathrm{d}r \end{aligned}$$

$$+\sum_i c_i(r_0) Q_i^{-1} \mathscr{G}(\omega_i) + \sum_i c_i(r_0) \varepsilon_i \, , \qquad (47)$$

where for simplity I have used i to label the modes rather than nl. Here ε_i are the errors in the observed $\delta\omega_i/\omega_i$, assumed uncorrelated, with standard deviations σ_i. Also,

$$\mathscr{K}_{c^2,\rho}(r_0, r) = \sum_i c_i(r_0) K^i_{c^2,\rho}(r) \qquad (48)$$

is the so-called *averaging kernel*, usually normalized such that $\int \mathscr{K}_{c^2,\rho}(r_0, r) \mathrm{d}r = 1$, and

$$\mathscr{C}_{\rho,c^2}(r_0, r) = \sum_i c_i(r_0) K^i_{\rho,c^2}(r) \qquad (49)$$

is the *cross-term kernel*. The goal of the inversion procedure is to determine the *inversion coefficients* $c_i(r_0)$, either explicitly or implicitly, such that $\mathscr{K}(r, r_0)$ is localized near $r = r_0$ and the remaining terms on the right-hand side of Eq. (47) are small. In this case the combination of the data provides a localized average of the sound-speed correction $\delta_r c^2/c^2$ near r_0.

The inversion can be carried out through a least-squares fit of parametrized versions of the functions $\delta_r c^2/c^2$, $\delta\rho/\rho$ and $\mathscr{G}$ in Eq. (35), using suitable regularization to reduce the errors in the solution and suppress unphysical variations (*e.g.* Dziembowski *et al.* 1990). From this fit the inversion coefficients can be determined (for the simpler case of rotational inversion, see for example Christensen-Dalsgaard, Schou & Thompson 1990). However, most inversions for solar structure have used the so-called optimally localized averages techniques, where the coefficients $c_i(r_0)$ are explicitly determined. As an example, I consider the Subtractive Optimally Localized Averages (SOLA) technique, originally developed by Pijpers & Thompson (1992, 1994) for rotational inversion. Here the coefficients are determined by matching $\mathscr{K}_{c^2,\rho}(r, r_0)$ to a suitably chosen target function $\mathscr{T}(r, r_0)$, while keeping the other contributions small. Specifically, one minimizes

$$\int_0^R \left[\mathscr{K}_{c^2,\rho}(r_0, r) - \mathscr{T}(r_0, r) \right]^2 \mathrm{d}r + \beta \int_0^R \mathscr{C}_{\rho,c^2}(r_0, r)^2 \mathrm{d}r + \mu \sum_i \sigma_i^2 c_i(r_0)^2 \, , \qquad (50)$$

subject to

$$\sum_i c_i(r_0) Q_i^{-1} \psi_\lambda(\omega_i) = 0 \, , \lambda = 0, \ldots, \Lambda \, , \qquad (51)$$

for a suitably chosen set of functions ψ_λ. In the expression (50) the first term serves to ensure that $\mathscr{K}$ is close to $\mathscr{T}$ while the second and third term control the contribution from $\delta\rho/\rho$ and the observational errors to the solution. The relative weight given to these terms is controlled by the *trade-off* parameters β and μ: if they are increased, the contributions from the unwanted terms are reduced, albeit generally at the expense of the match to the target function or more generally the resolution in the inversion. The additional constraints in Eqs (51) serve to suppress the surface contributions (*e.g.* Däppen *et al.* 1991; Kosovichev *et al.* 1992): since $\mathscr{G}$ is generally a slowly varying function of frequency the contribution is suppressed if the ψ_λ, *e.g.* chosen to be low-order polynomials, span such functions. This technique is generally successful in providing localized information about solar structure in most of the solar interior, with the exception of the innermost 5 – 10 per cent of the radius and the region very near the surface. Since the radial wavelength of the eigenfunctions, and hence in some sense the resolution of the kernels, scale as the sound speed c (*e.g.* Thompson 1993), it is common to choose the target functions as Gaussians with a width proportional to c. The determination of the scaling factor, and the other parameters (β and μ) characterizing the inversion is an important part of the procedure (for details, see for example Rabello-Soares, Basu & Christensen-Dalsgaard 1999).

The SOLA technique can be immediately generalized to determine, for example, the intrinsic correction $(\delta\gamma_1/\gamma_1)_{\rm int}$ from Eqs (36) (Basu & Christensen-Dalsgaard 1997). An important limitation for the investigations of the equation of state, however, is the assumption that the surface term is just a function of frequency; this breaks down for the high-degree modes that are particularly interesting for the study of the ionization zones of hydrogen and helium (*e.g.* Antia 1995; Gough & Vorontsov 1995). A technique to include corrections to the surface term in the inversion was developed by Di Mauro *et al.* (2002); with modes of degree as high as 1000 it is possible to obtain high resolution throughout the ionization zones of helium (see also Rabello-Soares *et al.* 2000).

4. OBSERVATION OF SOLAR OSCILLATIONS

Solar oscillations have been observed with high precision using a variety of techniques; a detailed review of observing and data analysis techniques was given by Brown (1996). The most extensive data have been obtained by means of observations of the line-of-sight velocity, using the Doppler shift; with suitable instrumentation it is possible to obtain *Doppler images* of the solar surface, such that each point of the images provides a measure of the surface velocity. Very extensive data have been obtained with the GONG[8] 6-station network of observing stations, suitably placed around the World (Harvey *et al.* 1996), and from instruments on the SOHO[9] satellite, including the Michelson Doppler Imager (MDI) (Scherrer *et al.* 1995). Additional data of very high precision on low-degree modes have been obtained by observing the Sun as a star, in light integrated over the solar disk, with the BiSON (Chaplin *et al.* 1996) and IRIS (Fossat 1991) groundbased networks as well as with the GOLF instrument (Gabriel *et al.* 1997) on SOHO.

Modes are excited in the Sun at degrees ranging from 0 to more than 1000, and over a range of frequencies. Thus the observed signal is a superposition of literally millions of modes, each with a spatial dependence on the solar surface corresponding to the projection of the displacement (Eq. 24) on the line of sight and with different frequencies. To determine the individual frequencies these contributions must be separated. The first step of the analysis is to carry out a spatial projection of the signal with suitable weight functions, aiming to isolate the component corresponding to given spherical-harmonic parameters (l_0, m_0). Had observations been available over the entire solar surface this

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[8] Global Oscillation Network Group

[9] SOlar and Heliospheric Observatory

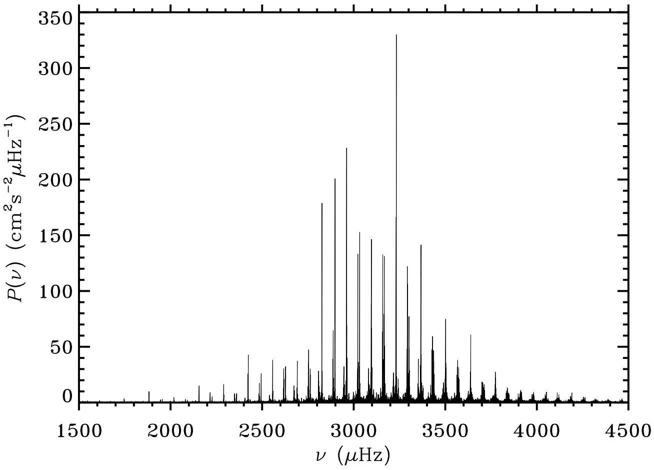

FIGURE 3. Power spectrum of solar oscillations, obtained from Doppler observations in light integrated over the disk of the Sun. The frequency is expressed in terms of cyclic frequency $\nu = \omega/2\pi$. The ordinate is normalized to show velocity power per frequency bin. The data were obtained from six observing stations and span approximately four months. (See Elsworth *et al.* 1995.)

could in principle have been achieved using a weight function based on $Y_{l_0}^{m_0}$, given the orthogonality of the spherical harmonics. With data for only slightly less than one solar hemisphere such complete separation is not possible and the result of the projection will contain components from other (l,m) in the vicinity of (l_0,m_0). By applying the projection to a series of Doppler images one obtains a timeseries which may then be Fourier analysed in time to isolate the individual modes.

The actual analysis is greatly complicated by the dense spectrum of modes, including the cross-talk between different spherical harmonics, and by the stochastic nature of the excitation and the finite lifetime of the modes. However, efficient and reliable techniques have been developed to determine the mode properties, and comparison of independent datasets and analysis methods have provided some confidence that reasonably consistent results on solar structure can be obtained (*e.g.* Basu *et al.* 2003). Even so, as discussed by Vorontsov (this volume) further work is needed to reduce remaining systematic errors in the results, which are becoming increasingly important as the helioseismic investigations are pushed towards ever finer aspect of the physics of the solar interior.

Some properties of the observed modes are illustrated in Figure 3, based on observations in integrated light with the BiSON network and hence restricted to modes of degree $l \leq 3$. The general distribution of mode power with frequency is common to all low and moderate degrees; the maximum amplitude for an individual mode is around $15\ \mathrm{cm\,s^{-1}}$ and modes have been detected with amplitudes as low as a few $\mathrm{mm\,s^{-1}}$. At low frequencies the natural linewidth of the modes is not resolved in these observations; here the lifetime is several months. At higher frequencies the lifetime is only a few days, and the finite width of the peaks in the spectrum is clearly visible.

Figure 4 show observed multiplet frequencies ν_{nl} obtained with the LOWL instrument (Tomczyk *et al.* 1995). The quality of the frequency determination is illustrated by the

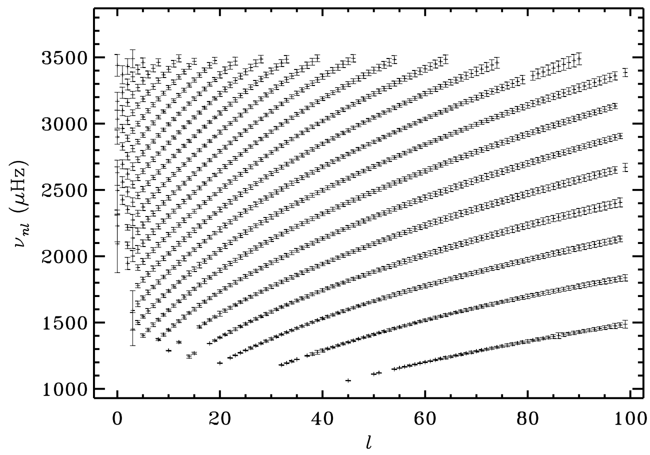

FIGURE 4. Observed solar p-mode frequencies, averaged over the azimuthal order *m*, as a function of the degree *l*, from one year of observations. The vertical lines show the 1000σ error bars. Each ridge corresponds to a given value of the radial order *n*, the lowest ridge having $n = 1$. (See Tomczyk, Schou & Thompson 1996).

fact that the data are shown with 1000σ error bars; the most accurately determined frequencies have a relative error of less than one part in 10^6, making them the most accurately known quantities related to the Sun. Evidently, it is this extremely high accuracy that allows detailed inferences to be made about the properties of the solar interior.

5. SOME RESULTS ON SOLAR STRUCTURE

The simplest analysis of the observed frequencies is evidently to compare them with frequencies of a solar model. Here I use as reference the so-called 'Model S' of Christensen-Dalsgaard *et al.* (1996). This is based on the Livermore equation of state (Rogers, Swenson & Iglesias 1996) and an early generation of the OPAL opacities (Iglesias, Rogers & Wilson 1992). Diffusion and settling of helium and heavier elements were included using the formulation of Michaud & Proffitt (1993). Figure 5a shows relative differences between the observed frequencies and those of Model S. To highlight the effect of near-surface errors, which are certainly present in the calculated frequencies, the differences have been scaled by the inertia ratio Q_{nl}; had the only error in the model been localized near the surface, the resulting scaled differences would have been solely a function of frequency. It is evident that, in fact, the strongest dependence is on frequency but with substantial variation also at given frequency. The nature of this variation is shown in panel (b), where a fitted function of frequency has been subtracted and the residuals plotted against $\nu/(l+1/2)$ or, as indicated, equivalently against the

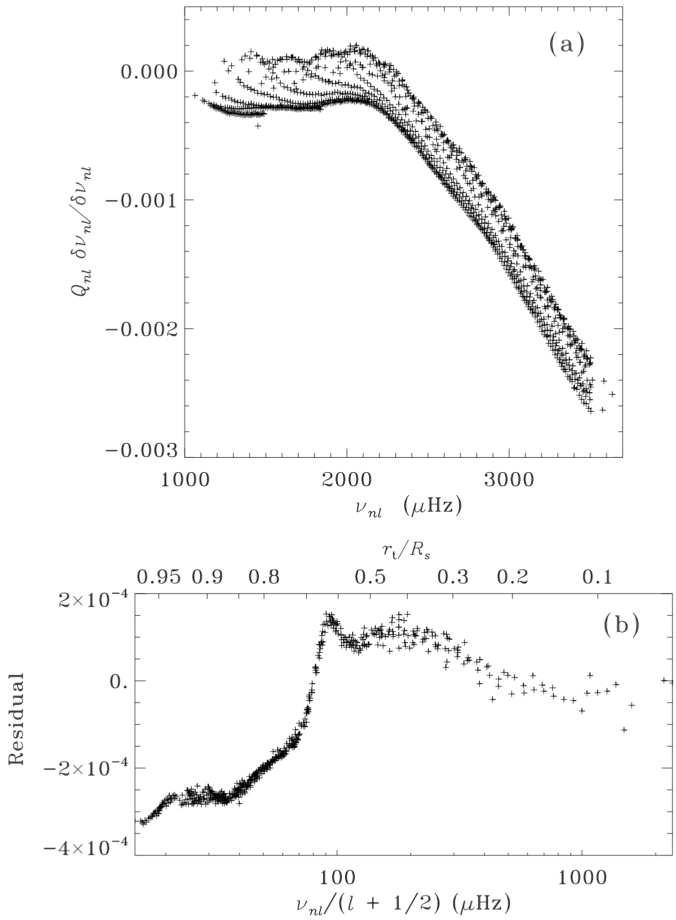

FIGURE 5. (a) Relative frequency differences, in the sense (observation) – (model), scaled by the inertia ratio Q_{nl} (*cf.* Eq. 35). The observations are a combination of BiSON whole-disk measurements (*e.g.*, Elsworth *et al.*, 1994) and LOWL observations (Tomczyk *et al.*, 1995), as described by Basu *et al.* (1997), while the computed frequencies are for Model S. (b) Scaled differences after subtraction of a fitted function of frequency, plotted against $\nu_{nl}/(l+1/2)$ which determines the inner turning point $r_{\rm t}$, shown as the upper abscissa.

location $r_{\rm t}$ of the inner turning point (*cf.* Eq. 39).[10] It should be noted that the residuals are highly systematic, with little scatter; this once more illustrates the extremely small random errors in the observed frequencies.

The separation of the differences is essentially equivalent to the asymptotic separation in Eq. (43) and thus the residuals can be identified with $\mathscr{H}_1$ (*cf.* Eq. 44), apart from an arbitrary constant shift. It is evident that the contribution for modes trapped within the convection zone, with $r_{\rm t} \gtrsim 0.7$, is modest, whereas clearly there is a drastic variation

[10] A more careful asymptotic analysis of low-degree modes shows that $\sqrt{l(l+1)}$ must be replaced by $l+1/2$, in Eq. (39) and the following equations.

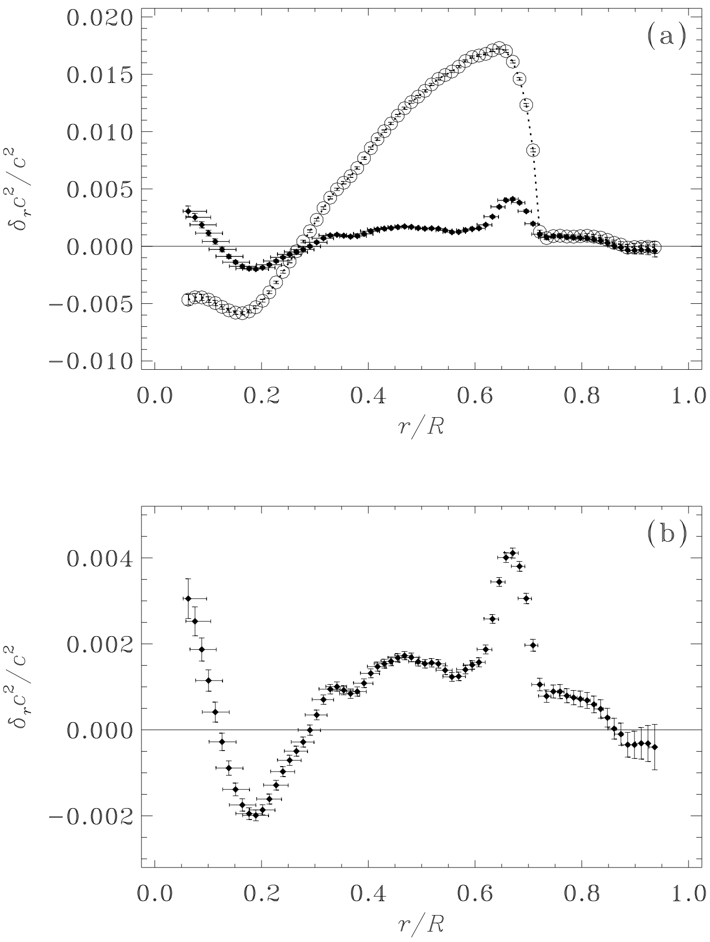

FIGURE 6. Results of sound-speed inversion. (a) Difference in squared sound speed, in the sense (Sun) – (model), inferred from inversion of the differences between the observed BiSON and LOWL frequencies and the frequencies of two solar models: closed circles are for Model S, and open circles for a similar model, but ignoring element diffusion and settling. (b) Results for Model S, on an expanded scale. The vertical error bars are 1-σ errors on the inferred differences, while the horizontal bars provide a measure of the resolution of the inversion. (Adapted from Basu *et al.*, 1997.)

just below this point, with further smaller effects contributing to even more deeply penetrating modes.

The detailed cause of this behaviour is revealed by inverse analysis of the frequency differences, based on the linearized relation (35) (see Section 3.4), to determine the relative difference in sound speed between the Sun and the model. The result is shown by the closed symbols in Fig. 6. It is evident that there is indeed a sharp feature in $\delta_r c^2/c^2$ just below the convection zone, with relatively modest variation elsewhere except in the core. Also, the overall magnitude of the differences is quite small, indicating that the model is a good approximation to the actual solar structure. On the other hand, the differences are evidently highly systematic and, given the inferred errors derived

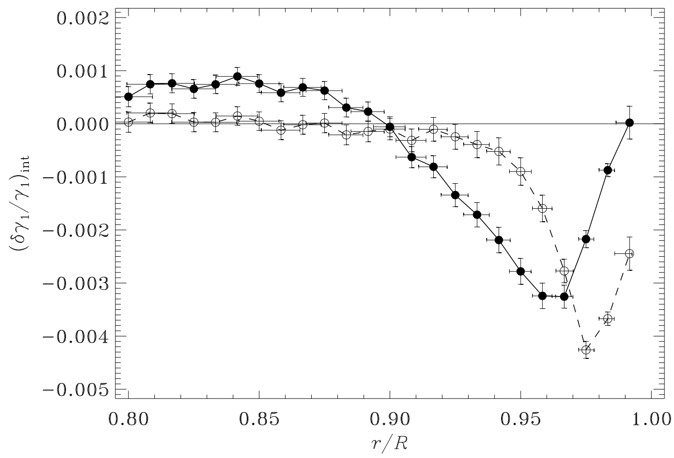

FIGURE 7. Relative difference between γ_1 obtained from an inversion of helioseismic data and γ_1 for two solar models, in the sense "Sun – model". Only the "intrinsic" difference in γ_1 is shown, that is, the part of the difference due to the equation of state (see text). Lines have been drawn through the results to guide the eye. The closed circles connected by a solid line are results obtained with an MHD model, the open circles connected with a dashed line are results with an OPAL model. (Adapted from Basu, Däppen and Nayfonov, 1999.)

from the quoted standard errors in the observations, highly significant. For comparison, panel (a) also shows results for a model that did not include diffusion and settling, but otherwise with the same physics; clearly the inclusion of diffusion and settling has resulted in major improvements in the agreement with solar structure.

The cause of the remaining differences between Model S and the Sun is obviously of great interest. It is striking that the largest differences occur in regions of strong variations in the hydrogen abundance X: just beneath the convection zone, where a sharp gradient is established by helium settling, and at the edge of the core where the gradient is set up by nuclear burning. In both cases partial mixing would change the composition and hence the sound speed in such a way as to improve the agreement between the model and the Sun. In the case of the bump just beneath the convection zone such mixing could result from overshoot from the convection zone or rotationally induced instabilities (*e.g.* Brun, Turck-Chièze & Zahn 1999; Elliott & Gough 1999; Brun *et al.* 2002). More generally, with suitable modification of the physics of the solar interior it is possible to produce 'seismic solar models' which closely match the structure inferred from helioseismology (*e.g.* Turck-Chièze *et al.* 2001; Couvidat, Turck-Chièze & Kosovichev 2003).

As discussed in connection with Eq. (36) it is possible to formulate the inverse problem for the solar frequencies in such a way as to allow a determination of the intrinsic error in γ_1. Results of such analyses are discussed extensively elsewhere in this volume; as an illustration, Fig. 7 shows the results for models using the MHD[11] and

[11] Mihalas, Hummer & Däppen; see, for example, Mihalas *et al.* (1990); Däppen (this volume).

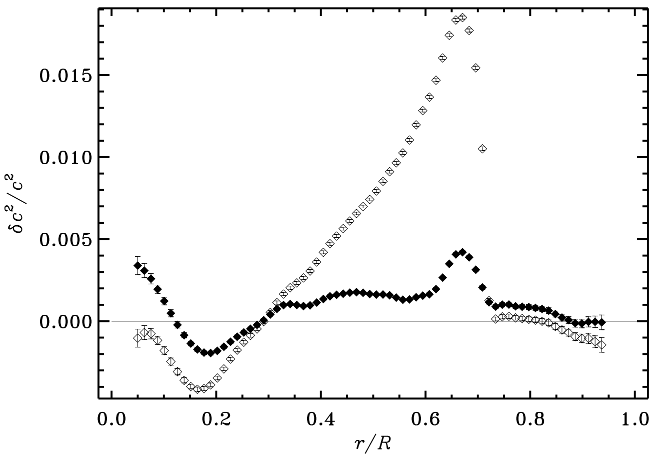

FIGURE 8. Effects of the revision of the inferred solar surface abundance on solar models. The filled symbols show the helioseismically inferred difference between squared sound speed in the Sun and in Model S, as in Fig. 6b, with $(Z/X)_s = 0.0245$. The open symbols show the corresponding results for a model approximately matching the revised abundances of Asplund *et al.* (2004) and consequently with $(Z/X)_s = 0.0183$. (Pijpers *et al.*, in preparation.)

the OPAL equations of state. It is evident that in most of the region shown, the error in γ_1 is smaller for the OPAL than for the MHD formulation, although with a possible reversal of this trend very near the surface. The results, including the formal error bars, are clear indications of the sensitivity of this type of helioseismic analysis to the thermodynamic properties of solar matter.

The results shown in Fig. 6 might motivate some complacency, or possibly pride, concerning our ability to model the Sun. After all, given the improvements in the physics of the solar interior up to the computation of Model S, but with no explicit adjustments of parameters, it was possible to match the solar sound speed to within a fraction of a per cent. It appears that such complacency would be misplaced. Recent spectroscopic analyses of the composition of the solar atmosphere, taking into account the three-dimensional dynamic nature of the atmosphere and departures from local thermodynamic equilibrium, have substantially decreased the inferred abundances of carbon, nitrogen and in particular oxygen (*e.g.* Allende Prieto, Lambert & Asplund 2002; Asplund *et al.* 2004). Since oxygen is a dominant source of opacity just below the convection zone, this leads to a corresponding reduction in the opacity and hence a change in the structure of solar models. In particular, the depth of the convection zone is reduced from $0.2885R$ in Model S, very close to the helioseismically inferred value of $0.287R$ (*e.g.* Christensen-Dalsgaard, Gough & Thompson 1991), to around $0.277R$. The effect on the comparison with helioseismic data is illustrated in Fig. 8, based on calculations by Pijpers *et al.* (in preparation). Owing to the reduction in the depth of the convection zone, the sound speed increases less rapidly with depth in the revised model, leading to the strongly enhanced peak in the sound-speed difference. The effects on the predicted solar neutrino flux were considered by Bahcall & Pinsonneault (2004). It was noted by Basu & Antia (2004) that a significant upward revision of the opacity tables, by more than 10 per cent, would be required to recover the agreement with the helioseismically inferred structure.[12] More detailed analyses of the required opacity changes have been carried out by Bahcall, Serenelli & Pinsonneault (2004) and Bahcall *et al.* (2004). Interestingly, Seaton & Badnell (2004) have found indications that the recent opacities from the Opacity Project are higher than the OPAL values generally used in current solar modelling, although possibly by an amount inadequate to account for the discrepancy between the Sun and the new models.

6. CONCLUDING REMARKS

It should be obvious that helioseismology has provided very detailed and precise information about the properties of the solar interior, including its thermodynamic state. This motivates its importance for the investigation of the equation of state of matter in near equilibrium under extreme conditions. It is also evident that the information so obtained is incomplete: the adiabatic oscillation frequencies do not yield information about the temperature or the internal energy of solar matter, although working in the nearly adiabatically stratified convection zone yields additional constraints that partly compensate for this. The information about the composition of solar matter is not very accurate; in the case of helium fairly precise constraints can be obtained from the oscillation frequencies, but only subject to uncertainties in the equation of state, making it difficult (and, at some level, impossible) to separate effects of errors in the helium abundance and the equation of state. Also, evidently, information is only available along the single (p, ρ) trace that is represented in the structure of the present Sun. Even so, the information obtained from the Sun clearly provide stringent constraints on the thermodynamical description and, as shown in Fig. 7, even sophisticated implementations of the equation of state fail to satisfy these constraints, to within a large margin.

In many ways the most interesting region, form a thermodynamic point of view, is the region relatively near the surface where ionization of hydrogen and helium takes place (*cf.* Fig. 1). Somewhat surprisingly, this region has been difficult to probe with helioseismology. To do so is greatly helped by data on high-degree modes, with turning points in the relevant region (*e.g.* Di Mauro *et al.* 2002). However, reliable determination of their frequencies is complicated by the fact that the individual modes cannot be separated, owing to cross-talk and the finite lifetime. As a result, different types of fit have to be developed. A detailed analysis of these problems, involving possibly also imperfections in the optics and the detectors, was presented by Korzennik, Rabello-Soares & Schou (2004). Also, the novel analysis techniques under development by Jefferies & Vorontsov (2004), discussed also by Vorontsov in this volume, are extremely promising.

Observations of other stars obviously allow to extend the space of parameters over

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[12] Such effects on solar structure of corrections to the opacity were discussed in considerable detail by Elliott (1995), who set up an inverse problem for the opacity errors, and Tripathy & Christensen-Dalsgaard (1998). Tripathy, Basu & Christensen-Dalsgaard (1998) attempted to estimate the opacity corrections required to match the differences between the Sun and Model S.

which the equation of state is investigated. For the foreseeable future such observations will be restricted to low-degree modes; however, even for these the observable effects of the sharp features associated with helium ionization may in principle be investigated (*e.g.* Pérez Hernández & Christensen-Dalsgaard 1998; see also Houdek, this volume), although data of the required quality are not imminent. Extreme conditions are found in white dwarfs, for which buoyancy driven oscillations (g modes) have been observed in many cases. Amongst hot white dwarfs cooling due to neutrino emission, certainly reflecting the thermodynamic state of their interiors, may be observable through period changes (*e.g.* O'Brien & Kawaler 2000); also, analysis of observed oscillation frequencies has provided some evidence for a crystallized core in a massive cool white dwarf (Metcalfe, Montgomery & Kanaan 2004). With further observations of the Sun and of this broad variety of stars there seems little doubt that our ability to test observationally the predictions of increasingly sophisticated treatments of the equation of state will expand greatly in the coming years.

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