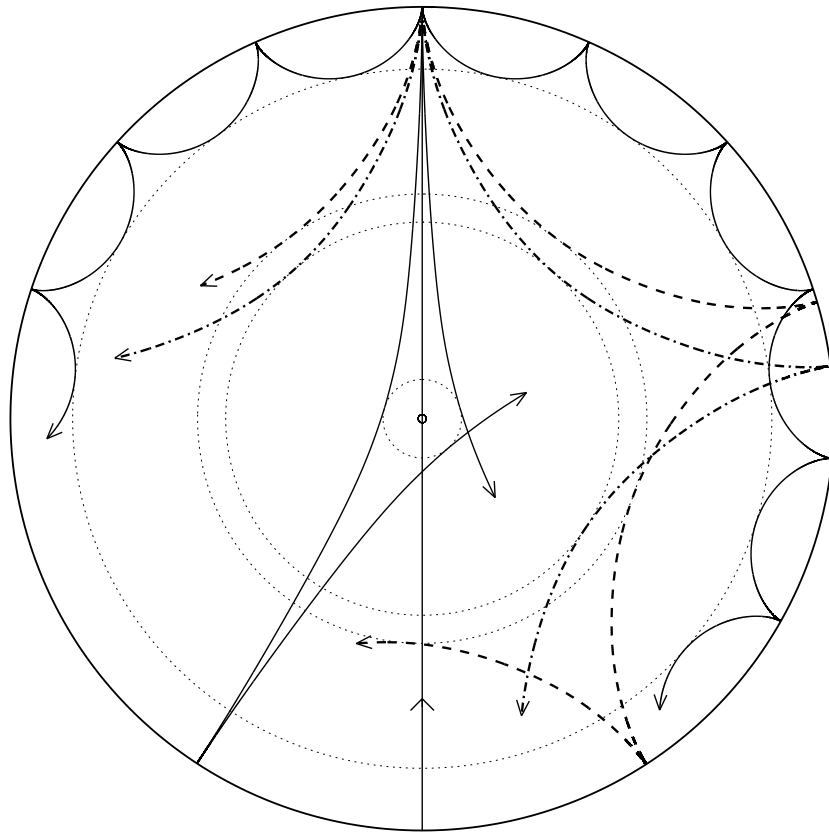


Lecture Notes on

# Stellar Oscillations

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*Cover:* Propagation of rays of sound waves through a cross section of a solar model (see also Section 5.2.3). The ray paths are bent by the increase with depth in sound speed until they reach the *inner turning point* (indicated by the dotted circles), where the waves undergo total internal refraction. At the surface the waves are reflected by the rapid decrease in density. The rays correspond to modes with frequency of  $3000 \mu\text{Hz}$ ; in order of decreasing depth of penetration their degrees  $l$  are: 0 (the straight ray passing through the centre), 2, 20, 25 and 75.

# Preface

The purpose of the present set of notes is to provide the technical background for the study of stellar pulsation, particularly as far as the oscillation frequencies are concerned. Thus the notes are heavily biased towards the use of oscillation data to study the interior of stars; also, given the importance of the study of solar oscillations, a great deal of emphasis is given to the understanding of their properties. In order to provide this background, the notes go into considerably more detail on derivations and properties of equations than is common, *e.g.*, in review papers on this topic. However, in a course on stellar pulsations they must be supplemented with other texts that consider the application of these techniques to, for example, helioseismology. More general background information about stellar pulsation can be found in the books by Unno *et al.* (1989) and Cox (1980). An excellent description of the theory of stellar pulsation, which in many ways has yet to be superseded, was given by Ledoux & Walraven (1958). Cox (1967) (reprinted in Cox & Giuli 1968) gave a very clear physical description of the instability of Cepheids, and the reason for the location of the instability strip.

The notes were originally written for a course in helioseismology given in 1985, and they were substantially revised in the Spring of 1989 for use in a course on pulsating stars.

I am grateful to the students who attended these courses for their comments. This has led to the elimination of some, although surely not all, errors in the text. Further comments and corrections are most welcome.

## Preface to 3rd edition

The notes have been very substantially revised and extended in this edition, relative to the previous two editions. Thus Chapters 6 and 9 are essentially new, as are sections 2.4, the present section 5.1, section 5.3.2, section 5.5 and section 7.6. Some of this material has been adopted from various reviews, particularly Christensen-Dalsgaard & Berthomieu (1991). Also, the equation numbering has been revised. It is quite plausible that additional errors have crept in during this revision; as always, I should be most grateful to be told about them.

## Preface to 4th edition

In this edition three appendices have been added, including a fairly extensive set of student problems in Appendix C. Furthermore, Chapter 10, on the excitation of oscillations, is new. The remaining revisions are relatively minor, although new material and updated results have been added throughout.

## Preface to 5th edition

The present edition has been extensively revised. New material includes a presentation of the recent data on solar-like oscillations in distant stars, which mark the beginning of a new era of asteroseismology. Also, the discussion of asymptotic eigenfunctions of stellar oscillations, and of stochastic excitation of solar-like oscillations, has been substantially extended.

Unlike previous editions, the present one has been typeset using L<sup>A</sup>T<sub>E</sub>X, leading to substantial changes in appearance and changes to the equation numbering.

I am grateful to Ross Rosenwald for his careful reading of the 4th edition, which uncovered a substantial number of misprints, and to Frank Pijpers for comments on a draft of the present edition. I thank Sarbani Basu, Francois Bouchy, Bill Chaplin, Yvonne Elsworth, Hans Kjeldsen, Jesper Schou, and Steve Tomczyk for help with figures or other material.

The present edition has been made available on the World Wide Web, at URL <http://astro.phys.au.dk/~jcd/oscilnotes/>.

*Aarhus, 1 June, 2003*

*Jørgen Christensen-Dalsgaard*

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